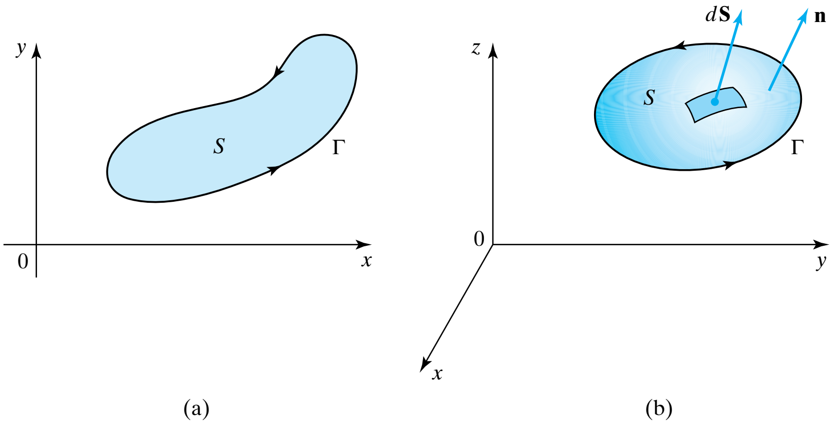


FIGURE 12.2 (a) A plane oriented surface. (b) A general oriented surface in space.

deriving Stokes' theorem it will be necessary to work with a two-sided open surface S bounded by a closed non-self-intersecting space curve Γ around which there is a given sense of direction. The normal at each point of S will be always be chosen in such a way that it points in the direction in which a right-handed screw would advance were it to be rotated in the sense of direction that is specified around the boundary curve Γ. Surfaces S of this type are called **oriented surfaces**. Pathological one-sided surfaces such as Möbius strips are said to be **nonorientable**, and they will not be considered here.

A simple but typical example of an open orientable surface S is an area in the (x, y)-plane contained within a closed curve Γ. If the sense of direction around Γ is chosen to be counterclockwise, the normal $\mathbf{n}$ to S will point in the direction of the unit vector $\mathbf{k}$. A reversal of the sense of direction around Γ will reverse the sense of $\mathbf{n}$, which will then point in the direction of $-\mathbf{k}$. Examples of *oriented surfaces* are illustrated in Fig. 12.2, where Fig. 12.2(a) shows an open oriented surface S in the (x, y)-plane and Fig. 12.2(b) shows a general open oriented surface in space.

Let S be a two-sided surface with a boundary curve Γ around which a sense of direction is prescribed, and at each point of S let $\mathbf{n}$ be the unit normal to S pointing in the direction determined by the sense of direction around Γ, as described above. Then if dS is an element of area of S, the vector element of area on the oriented surface S is $d\mathbf{S} = \mathbf{n}dS$.

Summary

This brief section introduced the important concept of an open surface that is orientable, and established the right-handed screw convention by which the direction of the normal to an orientable surface is determined.

12.2 Integral Theorems

The first integral theorem to be established is the Gauss divergence theorem, which relates volume integrals and surface integrals. It is possible to formulate a more general statement of the theorem than the one given here, but to do so involves a lengthy argument, and Theorem 12.1 is sufficient for all practical purposes.

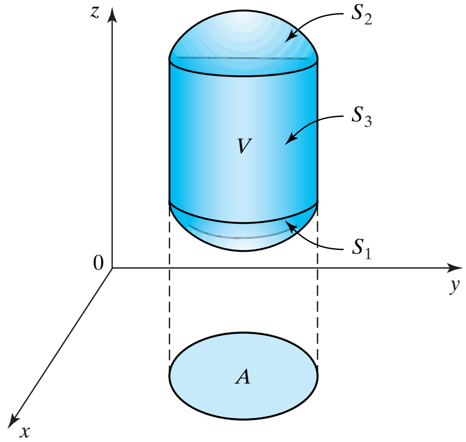


FIGURE 12.3 The volume V.

THEOREM 12.1

a theorem relating the integral of div F over a volume to the integral of the normal component of F over the surface bounding the volume

The Gauss divergence theorem Let $\mathbf{F}$ be a vector field defined throughout a volume V enclosed within a piecewise smooth surface S on which the outward drawn unit normal is $\mathbf{n}$. Then, if the components of $\mathbf{F}$ and its first order partial derivatives are continuous throughout V and on S, dV is an element of volume of V, and dS is an element of area of S,

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $d\mathbf{S} = \mathbf{n}\,dS$ is a vector surface element of area on S.

Proof Consider a volume V in the form of a cylinder with its sides parallel to the z-axis, a lower surface $z = z_1(x, y)$, and an upper surface $z = z_2(x, y)$, and let A be the projection of the cross-section of the cylinder onto the (x, y)-plane, as shown in Fig. 12.3.

The lower surface in Fig. 12.3 will be denoted by S_1, the upper surface by S_2, and the cylindrical side surface by S_3, so the surface S enclosing volume V is piecewise smooth and comprises these three surfaces.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, where the components of $\mathbf{F}$ and its first order partial derivatives are continuous in V and on S. The integral of $\partial F_3/\partial z$ with respect to z along a line in V drawn parallel to the z-axis is

$$\int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial F_3}{\partial z} dz = F_3(x, y, z_2(x, y)) - F_3(x, y, z_1(x, y)).$$

The integral of this result over the area A that is the projection of V onto the (x, y)-plane is given by

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_A F_3(x, y, z_2(x, y))dxdy - \iint_A F_3(x, y, z_1(x, y))dxdy.$$

The first term on the right is the integral of F_3 over the *top* of the upper two-sided surface S_2, while the second term is the integral F_3 over the *top* of the lower two-sided surface S_1. As the normals to surfaces bounding the volume V are chosen

to point *outward* from V, and the normal in the last term is directed *into* volume V, the sign of the last term can be reversed and the resulting equation written as

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_{S_2} F_3 dxdy + \iint_{S_1} F_3 dxdy.$$

To express the integrals on the right as a single integral over the complete surface S, it is necessary to take into account the integral of F_3 over the cylindrical surface S_3. The unit normal to the element of area $dxdy$ of A is perpendicular to the (x, y)-plane in the direction $\mathbf{k}$, but $\mathbf{k}$ is orthogonal to all outward drawn normals to the cylindrical surface, so the integral of F_3 over the cylindrical surface S_3 must vanish, giving $\iint_{S_3} F_3 dxdy = 0$. Adding this integral to the preceding equation, and recognizing that the piecewise smooth surface S comprises the sum of the three surfaces S_1, S_2, and S_3, we arrive at the result

$$\iiint_V \frac{\partial F_3}{\partial z} dV = \iint_S F_3 dxdy.$$

Corresponding results involving F_1 and F_2 that can be derived in similar fashion are

$$\iiint_V \frac{\partial F_1}{\partial x} dV = \iint_S F_1 dydz$$

and

$$\iiint_V \frac{\partial F_2}{\partial y} dV = \iint_S F_2 dxdz.$$

Addition of these three integrals gives

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \iint_S F_1 dydz + F_2 dxdz + F_3 dxdy,$$

or equivalently,

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S F_1 dydz + F_2 dxdz + F_3 dxdy.$$

Let dS with the outward drawn unit normal $\mathbf{n}$ be an element of area of the bounding surface S, and let its projection onto the (y, z)-plane be the element of area $dydz$. Then if the angle between $\mathbf{n}$ and the normal to the (y, z)-plane is γ, it follows that $dydz = dS \cos\gamma$. However, the unit normal to the (y, z)-plane is the vector $\mathbf{i}$, so $\cos\gamma = \mathbf{i} \cdot \mathbf{n}$, and consequently $dydz = \mathbf{i} \cdot \mathbf{n} dS = \mathbf{i} \cdot d\mathbf{S}$. Similar arguments lead to the corresponding results $dxdz = \mathbf{j} \cdot d\mathbf{S}$ and $dxdy = \mathbf{k} \cdot d\mathbf{S}$.

Using these expressions in the preceding integral allows it to be written as

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot \mathbf{n} dS$$

or as

$$\iiint_V \operatorname{div} \mathbf{F}\, dV = \iint_S F \cdot d\mathbf{S},$$

and the theorem is proved for a volume V with sides parallel to the z-axis. ■

Modifications to the preceding form of argument that we will not detail show the theorem to be true for volumes V with boundaries formed by finitely many piecewise smooth parts, and also for boundaries on which the partial derivatives of

F_i are not differentiable at every point. The theorem remains true for domains such as a torus that have a more complicated shape. This follows because such domains can be subdivided into domains of the type covered by Theorem 12.1, and as the outward-drawn normals to each side of a dividing surface are oppositely directed, the integrals over the two sides of each such surface cancel, leaving only the integral over S of the component of $\mathbf{F}$ normal to S.

CARL FRIEDRICH GAUSS (1777–1855)
A German mathematician of truly outstanding ability who is universally regarded as the greatest mathematician of the nineteenth century. He ranks with Isaac Newton as one of the greatest mathematicians of all time. He was appointed to the directorship of the observatory in Göttingen and spent the remainder of his life there. His contributions spanned all aspects of mathematics and science, in addition to his interest in astronomy. He also made important contributions to number theory, algebra, and geometry.

The divergence theorem provides an alternative definition of div $\mathbf{F}$, because if the result of the theorem is divided by the volume V with bounding surface S over which integration is performed, and the limit is taken as $V \to 0$ about a fixed point P in space, we obtain

$$(\operatorname{div} \mathbf{F})_P = \lim_{V\to 0} \frac{1}{V} \iint_S \mathbf{F} \cdot d\mathbf{S}. \tag{1}$$

However, $\mathbf{F} \cdot d\mathbf{S} = \mathbf{F} \cdot \mathbf{n} dS$ and $\mathbf{F} \cdot \mathbf{n} = F_n$ is the component of $\mathbf{F}$ normal to dS, so $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is the *flux* of $\mathbf{F}$ across S at the point P. Consequently, $(\operatorname{div} \mathbf{F})_P$ is seen to be the flux of $\mathbf{F}$ per unit volume at P.

A physical interpretation of this last result is provided by the flow of a fluid with velocity $\mathbf{q}$, because

$$(\operatorname{div} \mathbf{q})_P = \lim_{V\to 0} \frac{1}{V} \iint_S \mathbf{q} \cdot d\mathbf{S} \tag{2}$$

an application to incompressible flow with sources and sinks

is seen to be the amount of fluid leaving an infinitesimal surface surrounding P in a unit time. If the fluid is **incompressible**, there can be no net flow either into or out of any volume, so in an incompressible fluid div $\mathbf{q} = 0$ throughout the fluid. If, however, there is a **source** of fluid at P causing fluid to flow into volume V and onward out of S, then $(\operatorname{div} \mathbf{q})_P$ will be positive, whereas if there is removal of fluid from volume V at P due to the presence of a **sink** at P, then $(\operatorname{div} \mathbf{q})_P$ will be negative. In a fluid that is **compressible**, div $\mathbf{q}$ may be either positive or negative at a point in the fluid without any source or sink being present.

Any vector $\mathbf{F}$ such that

$$\operatorname{div} \mathbf{F} \equiv 0 \tag{3}$$

a solenoidal vector

is said to be a **solenoidal** vector. So as div(curl $\mathbf{F}$) $\equiv 0$, it follows that provided $\mathbf{F}$ has continuous second order partial derivatives, the vector curl $\mathbf{F}$ is a solenoidal vector.

The following examples illustrate how the divergence theorem can be used to simplify the evaluation of integrals, though more important applications arise in the formulation and solution of partial differential equations.

EXAMPLE 12.1 Evaluate

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy$$

where S is a smooth surface bounding an arbitrary volume V.

Solution The integral can be written

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = 3x\mathbf{i} + 2y\mathbf{j} - 5z\mathbf{k}$. So as the conditions of Theorem 12.1 are satisfied and $\operatorname{div} \mathbf{F} = 0$, it follows from the divergence theorem that

$$\iint_S 3xdydz + 2ydxdz - 5zdxdy = \iiint_V \operatorname{div} \mathbf{F}\, dV = 0.$$

■

EXAMPLE 12.2 Evaluate

$$\iint_S x^3dydz + y^3dxdz + z^3dxdy,$$

where the surface S is the boundary of the volume V occupying the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and above the plane $z = 0$.

Solution The volume V is a hemispherical shell between spheres of radii 1 and 2 centered on the origin and above the plane $z = 0$, so its surface S is formed by the surfaces of two hemispheres above the $z = 0$ plane and the annulus $1 \le r \le 2$ in the plane $z = 0$. The required integral can be written

$$I = \iint_S x^3dydz + y^3dxdz + z^3dxdy = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. As $\mathbf{F}$ is differentiable and the surface S is piecewise smooth, the divergence theorem can be used to replace the surface integral by the triple volume integral of $\operatorname{div} \mathbf{F}$ over V, showing that

$$I = 3\iiint_V (x^2 + y^2 + z^2)dxdydz.$$

The spherical symmetry of volume V suggests that integral I will be simplified if spherical polar coordinates are used. In terms of these coordinates, the volume V becomes $1 \le r \le 2, 0 \le \phi < 2\pi$, and $0 \le \theta \le \pi/2$, and the integrand becomes $x^2 + y^2 + z^2 = r^2$, so as the volume element of the transformation is given by $dV = r^2 \sin \phi dr d\theta d\phi$, the integral for I becomes

$$\begin{aligned} I &= 3\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_1^2 r^4 \sin\theta dr \\ &= 3\int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{31}{5} \sin\theta d\theta \\ &= \frac{93}{5}\int_0^{2\pi} d\phi = \frac{186}{5}\pi. \end{aligned}$$

■

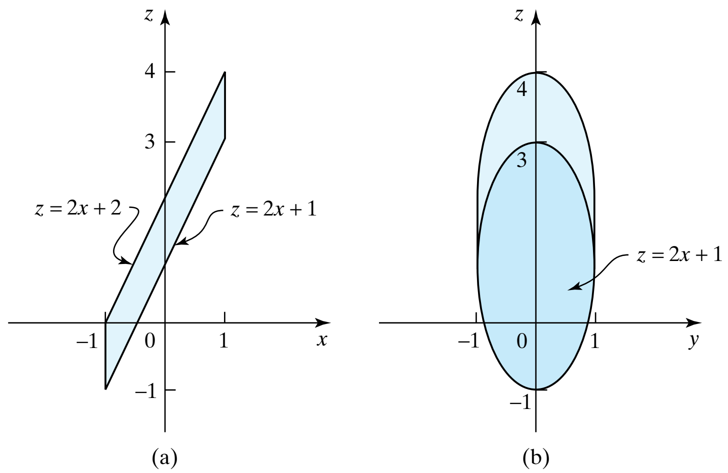


FIGURE 12.4 Cylinder with parallel oblique ends. (a) Side view; (b) front view.

EXAMPLE 12.3

Let the vector function $\mathbf{F} = (x^2 + 3y)\mathbf{i} - (3y^2 + \sin z)\mathbf{j} + 2z^2\mathbf{k}$ be defined throughout the volume V interior to the cylindrical volume with parallel oblique ends bounded by the surface S that is shown in Fig. 12.4, where the cylinder cross-section has the equation $x^2 + y^2 = 1$ and the cylinder ends are formed by the intersection of the cylinder with the planes $z = 2x + 1$ and $2x + 2$. Find the integral over S of F_n, the component of $\mathbf{F}$ normal to the surface S.

Solution The function $\mathbf{F}$ and the surface S satisfy the conditions of the divergence theorem, so as div $\mathbf{F} = 2x - 6y + 4z$, the result of applying the theorem to volume V is

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_V (2x - 6y + 4z)dV \\ &= \iint_{x^2+y^2\le 1}\left(\int_{1+2x}^{2+2x}(2x - 6y + 4z)dz\right)dxdy \\ &= \iint_{x^2+y^2\le 1}(10x - 6y + 6)dxdy.\end{aligned}$$

To proceed further, we change to plane polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ for which the Jacobian $J(r,\theta) = r$, and the area $x^2 + y^2 \le 1$ becomes $0 \le r \le 1$ with $0 \le \theta \le 2\pi$. As a result,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi} d\theta \int_0^1 (10r\cos\theta - 6r\sin\theta + 6)r\,dr \\ &= \int_0^{2\pi}\left(\frac{10}{3}\cos\theta - 2\sin\theta + 3\right)d\theta = 6\pi,\end{aligned}$$

so the required integral over S of the component F_n of $\mathbf{F}$ normal to S is

$$\iint_S \mathrm{F}_n dS = 6\pi.$$

■

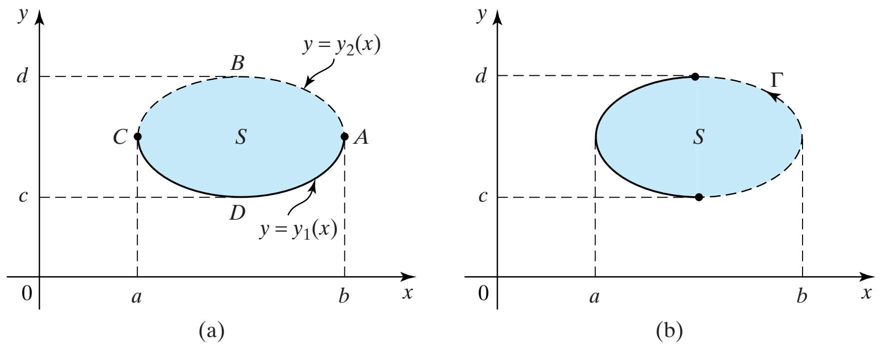


FIGURE 12.5 (a) The convex area S with lower and upper boundaries $y = y_1(x)$ and $y = y_2(x)$. (b) The convex area S with left and and right boundaries $x = x_1(y)$ and $x = x_2(y)$.

Preparatory to proving Stokes' theorem, we must prove Green's theorem in the plane that can be stated as follows.

THEOREM 12.2

a theorem relating an integral over a plane surface to an integral around its perimeter

Green's theorem in the plane Let a finite area S in (x, y)-plane be bounded by a piecewise smooth closed non-self-intersecting plane curve Γ around which a counterclockwise sense of direction is imposed. Then if $P(x, y)$ and $Q(x, y)$ and their first order partial derivatives are continuous over S and on Γ,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma Pdx + Qdy.$$

Proof We first prove the theorem for a plane area S that is convex, which is an area S with the property that any straight line that crosses it intersects the boundary at most twice. We then show how the theorem can be applied to more complicated areas, including those with internal boundaries. A typical area S of this type is shown in Fig. 12.5.

Let us consider the integral of $\partial P/\partial y$ over the convex area S with the lower boundary $y = y_1(x)$ and upper boundary $y = y_2(x)$, as shown in Fig. 12.5(a). The integral over S can be written as the iterated integral

$$\iint_S \frac{\partial P}{\partial y} dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy$$

$$= \int_a^b P(x, y_2(x))dx - \int_a^b P(x, y_1(x))dx$$

or as

$$\iint_S \frac{\partial P}{\partial y} dxdy = -\int_{ABC} P(x, y)dx - \int_{CDA} P(x, y)dx,$$

where the sign of the first integral on the right has been reversed because integration from $x = a$ to $x = b$ is in the opposite sense to the counterclockwise direction of integration required along ABC. The two arcs ABC and CDA form the closed

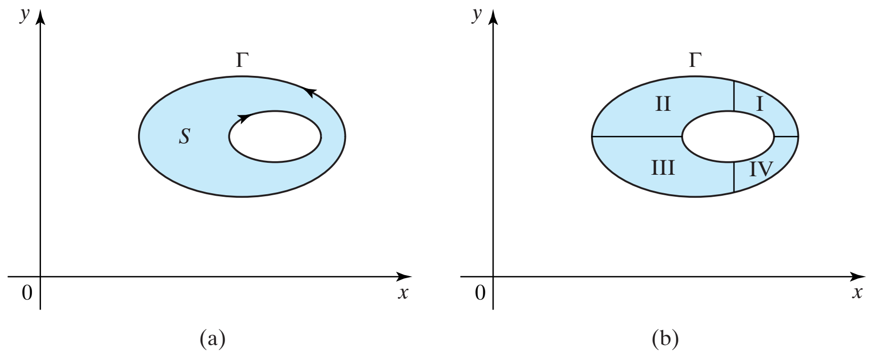


FIGURE 12.6 (a) S with an internal boundary. (b) The partitioning of S.

contour Γ, so the preceding result simplifies to

$$\iint_S \frac{\partial P}{\partial y} dxdy = -\int_\Gamma P(x, y)dx.$$

When the foregoing argument is repeated, but this time using the left and right boundaries in Fig. 12.5(b), and the integral of $\partial Q/\partial x$ over S is calculated we obtain

$$\iint_S \frac{\partial Q}{\partial x} dxdy = \int_\Gamma Q(x, y)dx.$$

However, as S is convex, each of these results is true, so subtracting them we arrive at the statement of Green's theorem

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma Pdx + Qdy.$$

We need to show this result remains true for areas S that are not convex, and also for areas with internal boundaries. It will be sufficient to consider the area S shown in Fig. 12.6(a), in which there is a single internal boundary γ, because the argument extends immediately to arbitrary areas with finitely many internal boundaries, and to areas that are not convex.

Let S be partitioned into the four areas shown in Fig. 12.6(b), to each of which Green's theorem applies. Applying the theorem to each area and adding the integrals, we see that integrals along the adjacent straight line segments will cancel, because of the continuity of P, Q, and their first order partial derivatives in S, and the fact that the integrations take place in *opposite* directions. As a result only the integrals around the boundaries Γ and γ remain, so the theorem holds, provided the sense of integration around all boundaries (both external and internal) is such that the area S always lies to the *left* as each boundary is traversed. This argument also applies to finitely many internal boundaries, so Green's theorem in the plane is proved for this more general case. ■

The sense in which integration must be performed when applying Green's theorem to an area S with internal boundaries is illustrated in Fig. 12.7.

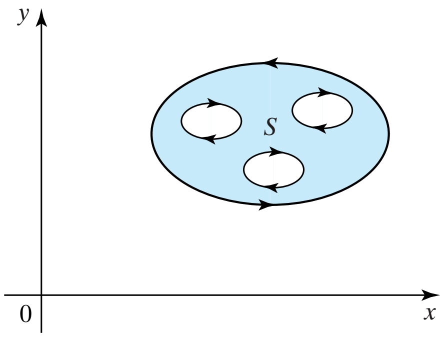


FIGURE 12.7 Direction of integration around a domain D with internal boundaries.

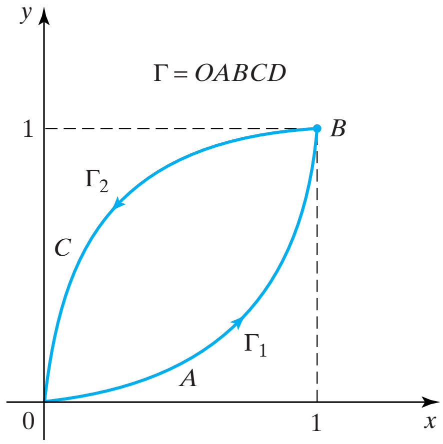


FIGURE 12.8 The curve Γ formed from two circular arcs Γ_1 and Γ.

GEORGE GREEN (1793–1841)
A self-taught English mathematical physicist who was born in Nottingham where he first worked as a baker. His contributions to electricity and magnetism, where he introduced the theorems now named after him, were first published privately in 1828, and so attracted little attention. It was not until William Thompson (Lord Kelvin) discovered his results and caused them to be republished in 1846 that their significance was recognized. Due to the limited circulation of the first published version of his work his main results were rediscovered, independently, by Lord Kelvin, Gauss, and others. He made significant contributions to the theory of optics and sound waves, and just prior to his death he was elected to a fellowship of Caius College, Cambridge.

SIR GEORGE GABRIEL STOKES (1819–1903)
A major applied mathematician and physicist who was born in County Sligo, Ireland, but spent his entire working life in Cambridge, where he was made professor of mathematics in 1849. He made fundamental contributions to the study of the flow of viscous fluids, leading to what are now called the Navier–Stokes equations, to elasticity, the propagation of sound, optics, and asymptotic series.

EXAMPLE 12.4 Evaluate

$$\int_{\Gamma} xy^2 dx - 2x^2 y dy$$

where Γ is the curve shown in Fig. 12.8, in which Γ_1 is an arc of a unit circle centered on the point $(0, 1)$, and Γ_2 is an arc of a unit circle centered on the point $(1, 0)$, and integration is in the counterclockwise sense around Γ.

Solution The equation of a unit circle with its center at $(1, 0)$ is $x^2 + (y-1)^2 = 1$, so the equation of the arc Γ_1 is $y = 1 - 1\sqrt{1-x^2}$ for $0 \leq x \leq 1$. The equation of a

unit circle with its center at (1, 0) is $(x-1)^2 + y^2 = 1$, so the equation of arc Γ_2 is $y = \sqrt{2x - x^2}$ for $0 \le x \le 1$.

Making the identifications $P = xy^2$ and $Q = -2x^2y$ we have $\partial P/\partial y = 2xy$ and $\partial Q/\partial x = -4xy$, so substituting into Green's theorem shows that

$$\int_\Gamma xy^2 dx - 2x^2 y dy = \int_0^1 dx \int_{1-\sqrt{1-x^2}}^{\sqrt{2x-x^2}} (-6xy) dy$$
$$= \int_0^1 [-6x^2 + 6x - 6x\sqrt{1-x^2}]dx = -1. \quad ■$$

THEOREM 12.3

a theorem relating an integral of the normal component of curl F over an orientable surface to the line integral of F around its perimeter

Stokes' theorem Let S be an open piecewise smooth orientable surface bounded by a closed space curve Γ around which a sense of direction is specified. At every point of the surface, let the unit normal $\mathbf{n}$ to S point in the direction specified for orientable surfaces relative to the sense around Γ. Then, if $\mathbf{F}$ is a differentiable vector function over the surface S,

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{r}$ is the position vector of a general point on Γ.

Proof Consider Fig. 12.9, in which S is an open orientable surface $z = z(x, y)$, Γ is its bounding space curve, A is the projection of S onto the (x, y)-plane, and C is the boundary curve of A.

The proof will involve the following three steps:

(I) The line integral around Γ will be transformed into the line integral around C

(II) The line integral around C will be transformed into a double integral over A

(III) The double integral over A will be transformed into an integral over S

STEP I Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then the line integral of F_1 around Γ is

$$\int_\Gamma F_1(x, y, z)dx = \int_C F_1(x, y, z(x, y))dx,$$

because $z = z(x, y)$ on C.

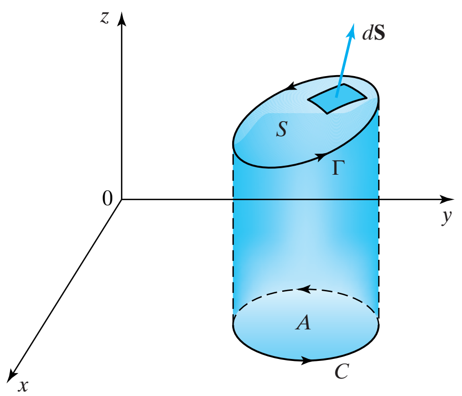


FIGURE 12.9 An orientable surface S bounded by the space curve Γ.

STEP II In the line integral on the right $z = z(x, y)$, so

$$\frac{\partial G_1}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}, \quad \text{where } G_1(x, y) \equiv F_1(x, y, z(x, y)).$$

Applying Green's theorem in the plane to the integral in Step I and using this last result gives

$$\int_C F_1(x, y, z(x, y))dx = -\iint_A \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}\right) dA,$$

where dA is the area element in the (x, y)-plane.

Setting $\phi = z - z(x, y)$, the surface S has the equation $\phi = 0$, so as a normal $\mathbf{N}$ to S is given by $\mathbf{N} = \operatorname{grad} \phi$

$$\mathbf{N} = \pm\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right).$$

For $\mathbf{N}$ to have the correct *upward* direction relative to S, as required by the sense of direction of integration around the oriented surface S, it is necessary that the z-component of $\mathbf{N}$ be positive. Consequently, if we take the positive sign, the unit vector $\mathbf{n}$ normal to S is

$$\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\mathbf{k},$$

where the direction cosines n_1, n_2, and n_3 are given by

$$n_1 = -\frac{\partial z}{\partial x}\bigg/|\mathbf{N}|, \quad n_2 = -\frac{\partial z}{\partial y}\bigg/|\mathbf{N}|, \quad n_3 = 1/|\mathbf{N}| \quad \text{with}$$

$$|\mathbf{N}| = \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)^{1/2}.$$

It now follows from these results that

$$\frac{\partial z}{\partial y} = -\frac{n_2}{n_3}.$$

If we substitute this expression for $\partial z/\partial y$ in the double integral over A, it becomes

$$\int_C F_1 dx = -\iint_A \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{n_2}{n_3}\right) dA.$$

STEP III If dA is the projection of dS onto the (x, y)-plane, we have $dA = n_3\, dS$, so the last result in Step II can be written as the double integral over S

$$\int_C F_1 dx = -\iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{n_2}{n_3}\right) n_3 dS$$
$$= \iint_S \left(\frac{\partial F_1}{\partial z}n_2 - \frac{\partial F_1}{\partial y}n_3\right) dS.$$

Similar arguments show that

$$\int_C F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial x}n_3 - \frac{\partial F_2}{\partial z}n_1\right) dS$$

and

$$\int_C F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y}n_1 - \frac{\partial F_3}{\partial x}n_2\right) dS.$$

Finally, the addition of these three integrals gives

$$\int_C F_1 dx + F_2 dy + F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) n_1 dS + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) n_2 dS + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) n_3 dS,$$

or equivalently,

$$\int_\Gamma F_1 dx + F_2 dy + F_3 dz = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dxdz + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy,$$

which is one form of Stokes' theorem. To arrive at the form given in the statement of the theorem it is only necessary to write $d\mathbf{S} = \mathbf{n}dS$, and then to recognize that

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},$$

for the integral to become

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}. \qquad \blacksquare$$

Stokes' theorem is a generalization of Green's theorem in the plane that was used in its proof, so it is to be expected that Stokes' theorem must reduce to Green's theorem in the plane when the surface S is an area in the (x, y)-plane. That this is the case can be seen by taking $\mathbf{F}$ to be only a function of x and y, so that $\mathbf{F} = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$, because then the first form of Stokes' theorem that was proved reduces to

$$\int_\Gamma F_1 dx + F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy,$$

and apart from a change of notation, this is the result of Theorem 12.2.

Stokes' theorem provides a physical interpretation of curl $\mathbf{F}$ that is most easily understood in the context of a fluid flow with $\mathbf{F}$ representing the fluid velocity vector. Consider a small disc of fluid of radius ρ centered at $\mathbf{r} = \mathbf{r}_0$, as shown in Fig. 12.10,

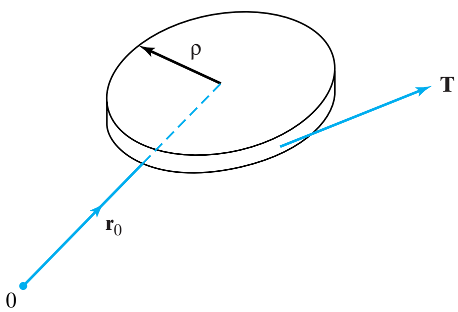


FIGURE 12.10 A disc of fluid of radius ρ with fluid velocity $\mathbf{F}$.

where S is the area of the disc and T is the unit tangent vector to the perimeter of the disc. Then $\mathbf{F} \cdot \mathbf{T}$ is the tangential component of the fluid velocity at the perimeter Γ of the disc around which the arc length is s, so the integral

$$\kappa(\mathbf{r}_0) = \int_\Gamma \mathbf{F} \cdot \mathbf{T} ds$$

is a measure of the tendency of the fluid to *rotate* around the point $\mathbf{r}_0$. This will be recognized as the *circulation* of $\mathbf{F}$ around a curve Γ introduced previously in connection with line integrals.

If the disc is small and taken on an open surface S in the fluid, and $\mathbf{N}$ is a unit normal to an element dS of the surface at $\mathbf{r} = \mathbf{r_0}$, the scalar product (curl $\mathbf{F}$) $\cdot$ $\mathbf{N}$ can be regarded as a constant over the disc, so from Stokes' theorem

$$\int_\Gamma \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{N} dS \approx [(\text{curl}\, \mathbf{F}) \cdot \mathbf{N}]_{\mathbf{r}_0} (\pi \rho^2),$$

and so

$$[(\text{curl}\, \mathbf{F}) \cdot \mathbf{N}]_{\mathbf{r}_0} = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \int_\Gamma \mathbf{F} \cdot \mathbf{T} ds.$$

Clearly, (curl $\mathbf{F}$) $\cdot$ $\mathbf{N}$ attains its greatest value when curl $\mathbf{F}$ is parallel to $\mathbf{N}$, and it is because curl $\mathbf{F}$ is a measure of rotation that some books use the notation rot $\mathbf{F}$ in place of curl $\mathbf{F}$. Although the circulation around Γ has been illustrated by means of a fluid flow, the general concept of the circulation of a vector $\mathbf{F}$ around a curve Γ has useful physical interpretations in other situations. Another example occurs in connection with the generation of current when a wire in the form of a closed curve Γ moves in a magnetic field. Inspection of the definition of (curl $\mathbf{F}$) $\cdot$ $\mathbf{N}$ at a point $\mathbf{r}_0$ as a limit shows it is the quotient of the circulation of $\mathbf{F}$ around Γ and the area of the disc, and so again measures the rate of circulation at $\mathbf{r}_0$.

EXAMPLE 12.5

Let $\mathbf{F} = x^2\mathbf{i} + z^2y\mathbf{j} + y^2z\mathbf{k}$. Show that the line integral of $\mathbf{F}$ around any space curve Γ bounding an oriented open surface S is zero.

Solution The conditions of Stokes' theorem apply and

$$\text{curl}\, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & yz^2 & y^2z \end{vmatrix} = \mathbf{0},$$

so

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = 0. \qquad \blacksquare$$

EXAMPLE 12.6

Let S be the surface of the paraboloid of revolution $z = 1 - x^2 - y^2$ with the domain of definition $x^2 + y^2 \leq 1$, and let Γ be the boundary of the paraboloid. Given $\mathbf{F} = x^3\mathbf{i} + (x + y - z)\mathbf{j} + yz\mathbf{k}$, find $\iint_S$ curl $\mathbf{F} \cdot d\mathbf{S}$.

Solution By Stokes' theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_\Gamma \mathbf{F} \cdot d\mathbf{r},$$

so the required integral can be found by evaluating the line integral on the right. As the domain of definition of the paraboloid of revolution is $x^2 + y^2 \le 1$, it follows that the curve Γ bounding the surface of the paraboloid is the circle $x^2 + y^2 = 1$ in the plane $z = 0$. To evaluate the line integral, we parametrize Γ as $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, with $0 \le t \le 2\pi$. Then $d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})dt$ and on Γ the vector function

$$\mathbf{F}(t) = \cos^3 t\mathbf{i} + (\cos t + \sin t)\mathbf{j},$$

so substituting into the line integral gives

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} [\cos^3 t\mathbf{i} + (\cos t + \sin t)\mathbf{j}] \cdot [-\sin t\mathbf{i} + \cos t\mathbf{j}]dt$$

$$\int_0^{2\pi} (-\sin t \cos^3 t + \cos^2 t + \sin t \cos t)dt = \pi.$$

EXAMPLE 12.7

Given $\mathbf{F} = y\mathbf{i} - z^3\mathbf{j} + x^2\mathbf{k}$, use Stokes' theorem to evaluate $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$, where Γ is the boundary of the area S formed by the part of the plane $2x + 4y + z = 4$ that lies in the first octant, and integration around the boundary Γ is in the clockwise direction.

Solution The required integral will be determined by evaluating the integral on the right of

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

The surface S over which integration is to be performed is the plane triangular area shown in Fig. 12.11, where the boundary of S in the plane $z = 0$ is the line $x + 2y = 2$

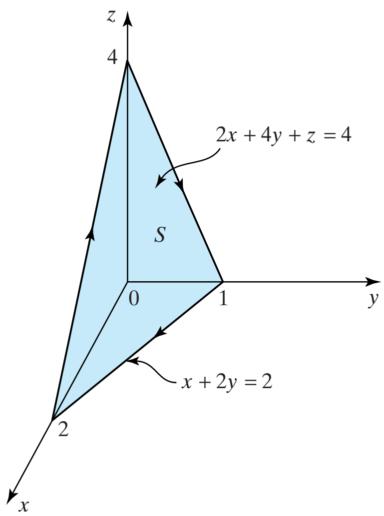


FIGURE 12.11 Plane triangular area S with clockwise direction around boundary Γ.

for $0 \leq x \leq 2$.

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -z^3 & x^2 \end{vmatrix} = 3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}.$$

If we set $\phi = 4 - 2x - 4y - z$, the equation of the plane is $\phi = 0$, so two possible normals $\mathbf{N}$ to the surface S of the plane are

$$\mathbf{N} = \pm \operatorname{grad} \phi = \pm(-2\mathbf{i} - 4\mathbf{j} - \mathbf{k}).$$

As the direction of integration around the boundary Γ is taken to be *clockwise*, when viewed as in Fig 12.11, the normal to S must be directed away from S toward the origin, showing that the $\mathbf{k}$ component of $\mathbf{N}$ must be *negative*. Thus, the foregoing expression for $\mathbf{N}$ must be chosen with the positive sign leading to the result $\mathbf{N} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, so the unit vector $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ with the required sense normal to the plane is

$$\mathbf{n} = \frac{1}{\sqrt{21}}(-2\mathbf{i} - 4\mathbf{j} - \mathbf{k}).$$

The line of intersection of the plane $2x + 4y + z = 4$ and the plane $z = 0$ is $x + 2y = 2$, so the base of the triangular plane surface S has the equation $x + 2y = 2$ for $0 \leq x \leq 2$.

We now have sufficient information to compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$:

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}) \cdot d\mathbf{S},$$

but $d\mathbf{S} = \mathbf{n}dS$, so if A is the projection of S onto the plane $z = 0$, the integral over S can be replaced by the integral over A, giving

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_S (3z^2\mathbf{i} - 2x\mathbf{j} - \mathbf{k}) \cdot d\mathbf{S} = \frac{1}{\sqrt{21}} \iint_S (-6z^2 + 8x + 1) dS.$$

However, if n_3 is the $\mathbf{k}$ component of $\mathbf{n}$, $dA/dS = |n_3| = 1\sqrt{21}$ and so $dS = \sqrt{21}dA$. Using this result in the integral on the right with $z = 4 - 2x - 4y$ shows that

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_A [-6(4 - 2x - 4y)^2 + 8x + 1] dA.$$

Writing the double integral over A as a repeated integral gives

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^1 dy \int_0^{-2-2y} [-6(4 - 2x - 4y)^2 + 8x + 1] dx = -\frac{29}{3}. \quad \blacksquare$$

The results of the next theorem, called **Green's formulas** or sometimes **Green's identities**, are used extensively in the study of partial differential equations.

THEOREM 12.4

Green's formulas Let Φ and Ψ be scalar fields such that the Laplacians $\Delta\Phi$ and $\Delta\Psi$ are defined inside a volume V enclosed in a closed piecewise smooth surface S, and if the second order partial derivatives of Φ and Ψ have any discontinuities, let them be bounded and occur only along lines on S or across finitely many surfaces in V. Then:

two useful formulas due to Green

(I) Green's first formula is

$$\iint_S \Phi \frac{\partial \Psi}{\partial n} dS = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV,$$

where dV is a volume element of V.

(II) Green's second formula is

$$\iint_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \iiint_V (\Phi \Delta \Psi - \Psi \Delta \Phi) dV.$$

Proof The proof is straightforward, but for simplicity it will only be offered for functions Φ and Ψ that have continuous second order partial derivatives inside a finite volume V and on its bounding surface S.

Setting $\mathbf{G} = \Phi(\text{grad } \Psi)$, it follows that

$$\text{div } \mathbf{G} = \Phi \, \text{div}(\text{grad } \Psi) + (\text{grad } \Phi) \cdot (\text{grad } \Psi),$$

so applying the divergence theorem we have

$$\iint_S \Phi(\text{grad } \Psi) \cdot d\mathbf{S} = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV.$$

However, $\Phi(\text{grad } \Psi) \cdot d\mathbf{S} = \Phi \mathbf{n} \cdot (\text{grad } \Psi) dS$, but $\mathbf{n} \cdot (\text{grad } \Psi)$ is simply the directional derivative of Ψ in the direction of the unit outward normal $\mathbf{n}$ that will be denoted by $\partial \Psi / \partial n$, so

$$\Phi(\text{grad}\Psi) \cdot d\mathbf{S} = \Phi \, \partial \Psi / \partial n dS.$$

Using this in the last result gives Green's first formula,

$$\iint_S \Phi \frac{\partial \Psi}{\partial n} dS = \iiint_V \{\Phi \Delta \Psi + (\text{grad } \Phi) \cdot (\text{grad } \Psi)\} dV.$$

Green's second formula follows directly from this by interchanging Φ and Ψ and subtracting the new result from the Green's first formula. ■

showing the uniqueness of the solution of $\Delta\phi = 0$ in a volume, on the surface of which ϕ is specified

In anticipation of Chapter 18, and as an illustration of the use of Green's first formula in the study of partial differential equations, we will prove the **uniqueness** of the solution ϕ of Laplace's equation

$$\Delta \phi = 0$$

in a volume V enclosed within a surface S on which the value of ϕ is specified at every point. Here, the *Laplacian* Δ can be considered to be expressed in terms of any system of orthogonal curvilinear coordinates, the simplest of which is, of course, the cartesian coordinate system where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

By the *uniqueness* of the solution of Laplace's equation, we mean that when ϕ is specified over the surface S enclosing a volume V, there is only *one* function ϕ that satisfies both Laplace's equation throughout V and the specified conditions for ϕ on the surface S. A typical physical example illustrating the interpretation of

this situation is provided by considering the steady state temperature distribution $T(x, y, z)$ throughout a cube of metal where the temperature is governed by the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

It is to be expected from a physical understanding of steady state heat conduction that the specification of a time-independent temperature distribution T over each face of the cube of metal will determine the temperature at each internal point of the metal, and that every time the surfaces of the same metal block are heated in the same way, the same internal temperature distribution will result. This is simply another way of saying that the solution of Laplace's equation subject to specified boundary conditions on S is expected to be *unique*.

The proof of this result is simple. Suppose, if possible, that two different solutions ϕ_1 and ϕ_2 exist that satisfy the *same* prescribed temperature conditions on S. Then, because Laplace's equation is linear, the function $\Phi = \phi_1 - \phi_2$ must also be a solution and, furthermore, $\Phi \equiv 0$ on S. Using this function Φ in Green's first formula and setting $\Psi = \Phi$ reduces it to

$$\iiint_D (\text{grad } \Phi) \cdot (\text{grad } \Phi) dV = 0.$$

The integrand is nonnegative, so this result can only be possible if grad $\Phi \equiv \mathbf{0}$, and this in turn implies that $\partial\Phi/\partial x = \partial\Phi/\partial y = \partial\Phi/\partial z = 0$, and so $\Phi =$ constant. However, as $\Phi = 0$ on the bounding surface S, this shows that $\Phi = 0$ throughout D, and so $\phi_1 \equiv \phi_2$ and the result is proved.

The theory and application of the vector integral calculus are developed in standard calculus and analytic geometry texts like those in references [1.1], [1.2], [1.5], [1.6], and [1.7]. More advanced and detailed accounts, with emphasis placed on a vector treatment, are to be found in references [5.1] to [5.3]. Extensive use of vector integral theorems in the study of hydrodynamics is made in reference [6.5].

Summary

The three fundamental integral theorems of Gauss, Green, and Stokes were proved, and in anticipation of the results of Chapter 18, a Green formula was used to establish the uniqueness of the solution of the Laplace equation $\Delta\phi = 0$ in a volume on the surface of which ϕ is specified. It will be seen later in Chapter 18 that this is called a *Dirichlet problem* for the Laplace equation, and it arises in many physical situations, such as the steady state temperature distribution in a solid, the electrostatic potential in a vacuum enclosed in a cavity, in problems of groundwater flow, and elsewhere.

EXERCISES 12.2

1. By setting $\mathbf{F} = \mathbf{a} \times \mathbf{G}$ in the divergence theorem, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{G}$ is a differentiable vector function defined in a volume V in a closed surface S, prove by using the properties of the scalar triple product that
$$\iint_S \mathbf{G} \times d\mathbf{S} = -\iiint_V \text{curl } \mathbf{G} dV.$$

2. Given a differentiable scalar function ϕ defined in a volume V contained in a closed surface S, prove that
$$\iiint_V (\text{grad } \phi) \times d\mathbf{S} \equiv \mathbf{0}.$$

3. Given the differentiable scalar and vector functions ϕ and $\mathbf{G}$, respectively, defined in a volume V in a closed

surface S, prove that

$$\iint_S \phi \mathbf{G}\cdot d\mathbf{S} = \iiint_V (\text{grad } \phi)\cdot \mathbf{G} dV + \iiint_V \phi \text{ div } \mathbf{G} dV.$$

4. Given the differentiable vector functions $\mathbf{P}$ and $\mathbf{Q}$ defined in a volume V bounded by a closed surface S, prove that

$$\iint_S \mathbf{P}\times\mathbf{Q}\cdot d\mathbf{S} = \iiint_V \mathbf{Q}\cdot \text{curl } \mathbf{P} dV - \iiint_V \mathbf{P}\cdot \text{curl } \mathbf{Q} dV.$$

5. The time-dependent heat equation can be written

$$\mu\rho\frac{\partial T}{\partial t} = \text{div}(\kappa \text{ grad } T),$$

where μ, ρ, and κ are material constants that may vary with position, t is the time, and T the temperature at a position $\mathbf{r}$ in a material occupying a volume V enclosed in a surface S. Prove that

$$\iint_S \kappa T(\text{grad } T)\cdot d\mathbf{S} = \iiint_V \kappa(\text{grad } T)\cdot(\text{grad } T)\, dV + \iiint_V \mu\rho T\frac{\partial T}{\partial t} dV.$$

6. Given that $\mathbf{R} = \text{curl } \mathbf{Q}$ and $\mathbf{Q} = \text{curl } \mathbf{P}$ are defined in a volume V enclosed in a surface S, prove that

$$\iiint_V \mathbf{Q}\cdot\mathbf{Q} dV = \iint_S \mathbf{P}\times\mathbf{Q}\cdot d\mathbf{S} + \iiint_V \mathbf{P}\cdot\mathbf{R} dV.$$

7. By using Stokes' theorem and considering curl $(\phi\mathbf{F})$, where ϕ and $\mathbf{F}$ are differentiable scalar and vector functions, respectively, both of which are defined over an open surface S with closed boundary curve Γ, prove that

$$\int_\Gamma \phi\mathbf{F}\cdot d\mathbf{R} = \iint_S (\text{grad } \phi)\times\mathbf{F}\cdot d\mathbf{S} + \iint_S \phi \text{ curl } \mathbf{F}\cdot d\mathbf{S}.$$

8. Given that ϕ and ψ are differentiable scalar functions defined over an open surface S with the closed boundary curve Γ, prove that

$$\int_\Gamma \phi(\text{grad } \Psi)\cdot d\mathbf{r} = \iint_S (\text{grad } \phi)\times(\text{grad } \psi)\cdot d\mathbf{S}.$$

9. Let $\mathbf{F} = -y^2\mathbf{i} + xz\mathbf{j} + z^2\mathbf{k}$ and S be the surface of the plane $x + y + 2z = 2$ lying in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ with a clockwise sense of direction around its triangular boundary Γ. Verify Stokes' theorem by computing $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$ and $\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$ and showing they are equal.

10. Given that $\mathbf{F} = yz\mathbf{i} + xy\mathbf{j} + x^2\mathbf{k}$ and S is the surface of the plane $x + 3y + z = 3$ lying in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ with a clockwise sense of direction around its triangular boundary $\Gamma$when seen from 0, verify Stokes' theorem by computing $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$ and $\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$ and showing they are equal.

12.3 Transport Theorems

In many applications the derivative with respect to time of surface and volume integrals is required where the integrand is a time-dependent field quantity and the surface or volume over which integration is to be performed moves with time. This situation arises, for example, when the rate of change of flux of a vector quantity $\mathbf{F}(\mathbf{r}, t)$ is required through an open surface $S(t)$ bounded by a moving closed space curve $\Gamma(t)$, or when the rate of change of a scalar quantity $f(\mathbf{r}, t)$ is required in a volume $V(t)$ that is enclosed in a moving surface $S(t)$. When computing the time derivative in the first case, it is necessary to take into account not only the time variation of the integrand, but also the effect of the moving boundary $\Gamma(t)$ of the surface $S(t)$ over which the time derivative of the flux is to be determined, whereas in the second case, in addition to the time dependence of $f(\mathbf{r}, t)$, the effect of the change in volume $V(t)$ must be considered.

Situations of this type occur when determining the generation of an electric current in a moving coil of wire in a magnetic field, in fluid mechanics when the energy content of a moving volume of fluid is considered and also in the study of shock waves, and in chemically reacting fluids where the chemical composition of a moving volume of fluid changes with time.

In this section two results called **transport theorems** will be derived. The first involves the rate of change of flux of a vector field across an open moving surface,

whereas the second concerns the rate of change of a volume integral of a scalar quantity when the volume involved is swept out by a moving open surface.

The first result involves computing the time derivative of the flux $\Phi(t)$ of a vector function $\mathbf{F}(\mathbf{r}, t)$ through an open surface $S(t)$ bounded by a closed time-dependent space curve $\Gamma(t)$. When deriving this result it will be assumed that the points on $S(t)$ and $\Gamma(t)$ move with a specified velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ that is defined throughout the region of space involved. The **flux** $\Phi(t)$ at time t is defined as the integral of the component of $\mathbf{F}(\mathbf{r}, t)$ normal to the surface $S(t)$, and so is given by

$$\Phi(t) = \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}, \tag{4}$$

where $d\mathbf{S}$ is an element of area of $S(t)$.

THEOREM 12.5

a transport theorem for the rate of change of flux

The flux transport theorem Let a vector field $\mathbf{F}(\mathbf{r}, t)$ be defined and differentiable in some region of space in which the points on an open surface $S(t)$ with a closed boundary curve $\Gamma(t)$ move with a prescribed velocity $\mathbf{q}(\mathbf{r}, t)$. Then the rate of change of the flux $\Phi(t)$ of the vector field $\mathbf{F}(\mathbf{r}, t)$ through $S(t)$ is given by

$$\frac{d\Phi}{dt} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\operatorname{div} \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r}.$$

Proof Consider the surface $S(t)$ at time t and the surface $S(t+h)$ at a subsequent time $t+h$ shown in Fig. 12.12, where the points of $S(t)$ move with the given velocity $\mathbf{v}(\mathbf{r}, t)$. Then $S(t)$ sweeps out the cylindrical volume $V(t)$ shown in the diagram, where the line AB on the side surface of the cylinder shows the path followed by point A on $\Gamma(t)$ as it moves to the corresponding point B on $\Gamma(t+h)$. Correspondingly, a typical point P on $S(t)$ will move to the point Q on $S(t+h)$ along the line PQ, where for a small time increment h the vector $\underline{AB} \approx \mathbf{v}(\mathbf{r}_A, t)h$, and the vector $\underline{PQ} \approx \mathbf{v}(\mathbf{r}_P, t)h$, where $\mathbf{r}_A$ and $\mathbf{r}_P$ are the position vectors of A and P.

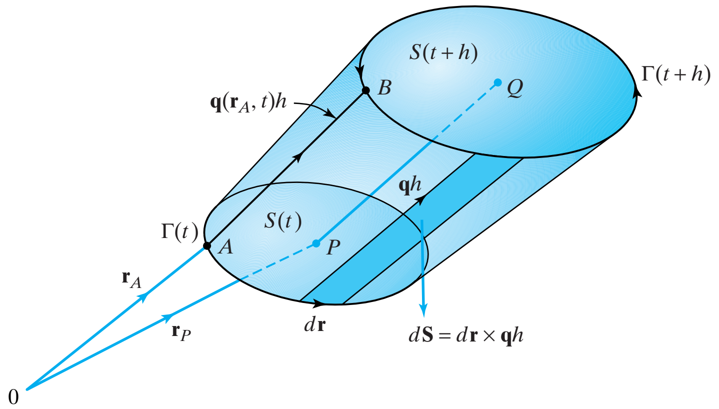


FIGURE 12.12 The surfaces S at times t and $t+h$ and the bounding curves $\Gamma(t)$ and $\Gamma(t+h)$.

It follows from the definition of a derivative that the time derivative of the flux $\Phi(t)$ is given by the limit

$$\frac{d\Phi}{dt} = \lim_{h \to 0} \left\{ \frac{1}{h} \left[\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} \right] \right\}. \tag{5}$$

In order to compute this limit, we first consider the difference

$$\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S},$$

and for small h use the Taylor approximation

$$\mathbf{F}(\mathbf{r}, t+h) \approx \mathbf{F}(\mathbf{r}, t) + h\frac{\partial \mathbf{F}}{\partial t}$$

to rewrite it as

$$\begin{aligned} &\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} \\ &\quad \approx \iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} + h \iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}. \end{aligned} \tag{6}$$

To proceed further, if V is the volume swept out by $S(t)$ in time increment h, then the *outward*-drawn normal to V at $S(t+h)$ is $d\mathbf{S}$, while the *outward*-drawn normal to V at $S(t)$ is $-d\mathbf{S}$. Denoting the side of the cylindrical volume by Σ and applying the divergence theorem to $\mathbf{F}(\mathbf{r}, t)$ in V gives

$$\iiint_V \operatorname{div} \mathbf{F}(\mathbf{r}, t) dV = \iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} + \iint_{\Sigma} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}. \tag{7}$$

Using (7) to eliminate $\iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}$ from (6) leads to the result

$$\begin{aligned} &\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} \\ &\quad \approx h \iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{S} + \iiint_V \operatorname{div} \mathbf{F}(\mathbf{r}, t) dV - \iint_{\Sigma} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S}. \end{aligned} \tag{8}$$

Now on the side Σ of the cylindrical surface the outward-drawn surface element $d\mathbf{S} = d\mathbf{r} \times \mathbf{q}h$, where $d\mathbf{r}$ is a vector element along $\Gamma(t)$ directed in the counterclockwise direction. The volume element dV swept out by $d\mathbf{S}$ in time increment h is the product of the area $|d\mathbf{S}|$ of $d\mathbf{S}$ and the perpendicular distance l between $S(t+h)$ and $S(t)$ given by $l = |\mathbf{q}h \cdot \mathbf{n}|$, where $\mathbf{n}$ is the unit normal to $d\mathbf{S}$, so that $dV = d\mathbf{S} \cdot \mathbf{q}h$. When these results are used to simplify (8) and h is small, it becomes

$$\begin{aligned} &\iint_{S(t+h)} \mathbf{F}(\mathbf{r}, t+h) \cdot d\mathbf{S} - \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} \\ &\quad \approx h \iint_{S(t)} \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{S} + h \iint_{S(t)} \operatorname{div} \mathbf{F}(\mathbf{r}, t)\mathbf{q} \cdot d\mathbf{S} + h \int_{\Gamma(t)} \mathbf{F}(\mathbf{r}, t) \times \mathbf{q} \cdot d\mathbf{r}, \end{aligned} \tag{9}$$

where the sign of the last term has been changed by using the result $\mathbf{F} \cdot d\mathbf{r} \times \mathbf{q} = -\mathbf{F} \times \mathbf{q} \cdot d\mathbf{r}$.

Using (9) in the difference quotient (5) and proceeding to the limit as $h \to 0$ brings us to the statement of the theorem:

$$\frac{d\Phi}{dt} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\operatorname{div} \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r}. \quad \blacksquare$$

EXAMPLE 12.8

Let $S(t)$ be a plane rectangular area with its corners at the points $(0, 0, z)$, $(x, 0, z)$, $(x, 1, z)$, and $(0, 1, z)$, where $x = vt$, $z = ut$, t is the time, and u and v are constant speeds. Verify the flux transport theorem in the case that $\mathbf{F} = xz\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

Solution To verify Theorem 12.5 it will first be necessary to compute $\Phi(t)$ in order to find $d\Phi/dt$ directly. The theorem will be verified in this case if this expression for $d\Phi/dt$ can be shown to equal the sum of the surface and line integrals on the right of the statement of the theorem when each has been computed separately.

The geometry of the problem is shown in Fig. 12.13(a), and the projection of $S(t)$ onto the (x, y)-plane is shown in Fig. 12.13(b). It can be seen from the statement of the problem that the rectangular area remains parallel to the (x, y)-plane while moving along the z-axis with the constant speed u, and that its length increases with constant speed v in the positive x-direction.

We have $\mathbf{F} = xz\mathbf{k}$, $z = ut$, $x = vt$, so as the motion is uniform in the x- and z-directions, each point of $S(t)$ must move with the velocity $\mathbf{q} = v\mathbf{i} + u\mathbf{k}$. The flux $\Phi(t)$ is given by

$$\Phi(t) = \iint_{S(t)} \mathbf{F}(\mathbf{r}, t) \cdot d\mathbf{S} = \int_0^1 \int_0^{vt} xz\mathbf{k} \cdot \mathbf{k}\, dx\, dy = \int_0^1 \int_0^{vt} xz\, dx\, dy.$$

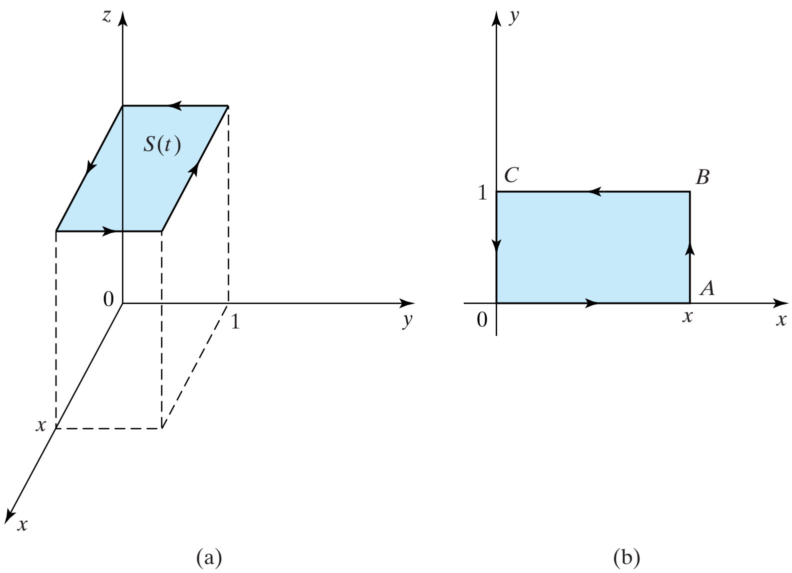


FIGURE 12.13 (a) The moving planar rectangle $S(t)$. (b) The projection of $S(t)$ onto the (x, y)-plane.

So as $z = ut$ is not involved in the integration, it can be removed as a factor to give

$$\Phi(t) = ut \int_0^1 \int_0^{vt} x dx dy = \frac{1}{2} uv^2 t^3,$$

so the rate of change of flux when computed directly is given by

$$\frac{d\Phi}{dt} = \frac{3}{2} uv^2 t^2.$$

Now $\partial \mathbf{F}/\partial t = \mathbf{0}$, div $\mathbf{F} = x$, and $d\mathbf{S} = dxdy\mathbf{k}$, so as

$$\left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + (\text{div } \mathbf{F})\right] \mathbf{q} = xv\mathbf{i} + xu\mathbf{k},$$

$$\iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + \text{div } \mathbf{F}\right] \mathbf{q} \cdot d\mathbf{S} = \int_0^1 \int_0^{vt} (xv\mathbf{i} + xu\mathbf{k}) \cdot \mathbf{k} dx dy$$

$$= u \int_0^1 \int_0^{vt} x dx dy = \frac{1}{2} uv^2 t^2.$$

A simple calculation shows that $\mathbf{F} \times \mathbf{q} = xvz\mathbf{j}$, and so

$$\int_\Gamma \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = \int_{\Gamma(t)} xvz\mathbf{j} \cdot d\mathbf{r} = uvt \int_{\Gamma(t)} x\mathbf{j} \cdot d\mathbf{r}.$$

Inspection of Fig. 12.13(b) shows that on OA, $d\mathbf{r} = dx\mathbf{i}$, on AB, $d\mathbf{r} = dy\mathbf{j}$, on BC, $d\mathbf{r} = -dx\mathbf{i}$, and on CO, $d\mathbf{r} = -dy\mathbf{j}$. The orthogonality of $\mathbf{i}$ and $\mathbf{j}$ means there are no contributions from the line integrals along OA and BC, and as $x = 0$ on OC there no contribution from the line integral along CO, so that

$$\int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = uvtx \int_0^1 dy = uv^2 t^2.$$

We see from this that

$$\iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial \mathbf{t}} + (\text{div } \mathbf{F})\mathbf{q}\right] \cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F} \times \mathbf{q} \cdot d\mathbf{r} = \frac{1}{2} uv^2 t^2 + uv^2 t^2 = \frac{3}{2} uv^2 t^2.$$

This result equals the expression for $d\Phi/dt$ found previously by direct computation, so the theorem has been verified in this case. ■

a theorem determining the rate of change of an integral over a volume $V(t)$ of a function of position and time when the surface bounding $V(t)$ is moving

The second transport theorem concerns the rate of change of a volume integral of a differentiable scalar function $f(\mathbf{r}, t)$ when the volume $V(t)$ over which integration is performed is bounded by a closed moving surface $S(t)$, so for this reason it is called the **volume transport theorem**. Because of the importance of this theorem in fluid mechanics, where it was first derived by Reynolds, it is also known as the **Reynolds transport theorem**.

THEOREM 12.6

The Reynolds transport theorem Let the scalar function $f(\mathbf{r}, t)$ be defined and differentiable in a region of space $V(t)$ through which the points inside and on a closed surface $S(t)$ move with a prescribed velocity $\mathbf{q}(\mathbf{r}, t)$. Then

$$\frac{d}{dt} \iiint_{V(t)} f(\mathbf{r}, t) dV = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{S(t)} f(\mathbf{r}, t)\mathbf{q} \cdot d\mathbf{S}.$$

OSBORNE REYNOLDS (1842–1912)
An Irish scientist and engineer, born in Belfast into a clerical family and educated in his early years by his father. After a year spent in the workshop of the inventor and mechanical engineer Edward Hayes he studied mathematics at Cambridge University and graduated in 1867. Shortly afterwards he was appointed to the newly established Chair of Engineering in Manchester University where he remained until his death. He made many important contributions to mechanical engineering and to fluid mechanics, where he introduced the nondimensional quantity (number) now called the Reynolds' number that determines when a fluid flow is smooth or turbulent. During his lifetime he received many awards.

Proof For simplicity we only offer an intuitive derivation of the theorem. Let a scalar function $f(\mathbf{r}, t)$ be defined and differentiable throughout some region in which a volume $V(t)$ enclosed in a closed surface $S(t)$ moves, and let the points of $V(t)$ and $S(t)$ move with a prescribed velocity $\mathbf{q}(r, t)$. Then our objective will be to compute

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV,$$

where dV is the volume element in $V(t)$. To accomplish this we start from the definition of a derivative in terms of a limit

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \lim_{h\to 0}\frac{1}{h}\left[\iiint_{V(t+h)} f(\mathbf{r}, t+h)dV - \iiint_{V(t)} f(\mathbf{r}, t)dV\right], \tag{10}$$

and write $V(t+h) = V(t) + \Delta(t, h)$, where $\Delta(t, h)$ represents the change in volume $V(t)$ in the time increment h. As a result of this (10) becomes

$$\begin{aligned}
&\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV \\
&= \lim_{h\to 0}\frac{1}{h}\left[\iiint_{V(t)} f(\mathbf{r}, t+h)dV - \iiint_{V(t)} f(\mathbf{r}, t)dV + \iiint_{\Delta(t,h)} f(\mathbf{r}, t)dV\right] \\
&= \lim_{h\to 0}\iiint_{V(t)}\frac{1}{h}[f(\mathbf{r}, t+h) - f(\mathbf{r}, t)]dV + \lim_{h\to 0}\frac{1}{h}\left[\iiint_{\Delta(t,h)} f(\mathbf{r}, t+h)dV\right] \\
&= \iiint_{V(t)}\frac{\partial f(\mathbf{r}, t)}{\partial t} + \lim_{h\to 0}\frac{1}{h}\left[\iiint_{\Delta(t,h)} f(\mathbf{r}, t+h)dV\right].
\end{aligned} \tag{11}$$

The volume $\Delta(t, h)$ is the change in volume of $V(t)$ in the time increment h, but in this time a surface element $d\mathbf{S}$ of $S(t)$ is displaced by the vector $\mathbf{q}h$, so the corresponding volume element swept out by $d\mathbf{S}$ in $\Delta(t, h)$ in this time interval is $dV \approx h\mathbf{q}\cdot d\mathbf{S}$. Consequently, (11) becomes

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \iiint_{V(t)}\frac{\partial f(\mathbf{r}, t)}{\partial t}dV + \lim_{h\to 0}\frac{1}{h}\left[\iint_{S(t)} hf(\mathbf{r}, t+h)\mathbf{q}\cdot d\mathbf{S}\right].$$

If we take the limit as $h \to 0$, when $f(\mathbf{r}, t+h) \to f(\mathbf{r}, t)$, this reduces to the statement of the theorem

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \iiint_{V(t)}\frac{\partial f}{\partial t}dV + \iint_{S(t)} f(\mathbf{r}, t)\mathbf{q}\cdot d\mathbf{S}.$$

■

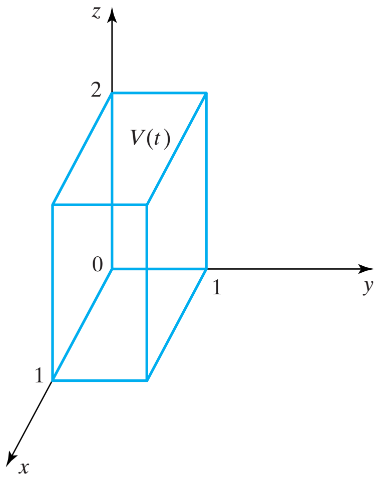


FIGURE 12.14 The rectangular parallelepiped with its top surface moving vertically with the constant speed u.

EXAMPLE 12.9

Verify the Reynolds transport theorem when $f = x^2yzt$ and the volume $V(t)$ is the rectangular parallelepiped with the corners of its base at the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$, its sides normal to the (x, y)-plane, and the corners of its upper surface at the points $(0, 0, z)$, $(1, 0, z)$, $(1, 1, z)$, and $(0, 1, z)$ when $z = ut$, with t the time and u a constant speed.

Solution The geometry of the problem is shown in Fig. 12.14. To verify the Reynolds transport theorem, it is necessary first to compute the integral $\iiint_{V(t)} f(\mathbf{r}, t)dV$, and then to find its derivative with respect to time t. The theorem will be verified if this result can be shown to equal the sum of the two integrals on the right of the theorem when they are evaluated separately:

$$\iiint_{V(t)} f(\mathbf{r}, t)dV = \int_0^1 \int_0^1 \int_0^{ut} x^2yzt\,dzdydx = \frac{1}{3}\frac{1}{2}\frac{1}{2}u^2t^2t = \frac{1}{12}u^2t^3,$$

so

$$\frac{d}{dt}\iiint_{V(t)} f(\mathbf{r}, t)dV = \frac{1}{4}u^2t^2.$$

We have

$$\iiint_{V(t)} \frac{\partial f}{\partial t}dV = \int_0^1 \int_0^1 \int_0^{ut} x^2yz\,dzdydx = \frac{1}{3}\frac{1}{2}\frac{1}{2}u^2t^2 = \frac{1}{12}u^2t^2,$$

and as $\mathbf{q} = u\mathbf{k}$ and $d\mathbf{S} = dxdy\mathbf{k}$,

$$\iint_{S(t)} f(\mathbf{r}, t)\mathbf{q}\cdot d\mathbf{S} = z\int_0^1 \int_0^1 x^2ytu\,dydx = \frac{1}{3}\frac{1}{2}u^2t^2 = \frac{1}{6}u^2t^2.$$

The theorem is verified, because $\frac{1}{12}u^2t^2 + \frac{1}{6}u^2t^2 = \frac{1}{4}u^2t^2$. ■

Summary

The flux transport theorem and the Reynolds' transport theorem, also known as the volume transport theorem, were proved and applied. Typical examples of the application of these theorems is the use of the first theorem to determine the rate of change of electric flux through a moving coil of wire in a generator, and the use of the second theorem when considering the continuity equation in fluid mechanics.

EXERCISES 12.3

1. Verify the rate of change of flux theorem given that $\mathbf{F} = xz\mathbf{k}$ and $S(t)$ is the plane rectangular surface with its corners at the points $(0, 0, z)$, $(x, 0, z)$, (x, y, z), and $(0, y, z)$, where $x = ut$, $y = vt$, and $z = wt$, with t the time and $u > 0$, $v > 0$, $w > 0$ a constant speed.

2. Verify the rate of change of flux theorem given that $\mathbf{F} = xz\mathbf{k}$ and $S(t)$ is the plane rectangular surface with its corners at the points $(0, 0, z)$, $(1, 0, z)$, $(1, y, z)$, and $(0, y, z)$, where $y = vt$ and $z = \alpha t^2$, with t the time and $v > 0$ a constant speed.

3.* A volume $V(t)$ in the form of a rectangular parallelepiped has the corners of its base at the points $(0, 0, z_1)$, $(1, 0, z_1)$, $(1, 1, z_1)$, and $(0, 1, z_1)$ with its sides perpendicular to the (x, y)-plane and the corners of its top surface at the points $(0, 0, z_2)$, $(1, 0, z_2)$, $(1, 1, z_2)$, and $(0, 1, z_2)$, where $z_1 = ut$ and $z_2 = vt$, with t the time and u, v constant speeds such that $u > 0$, $v > 0$. Verify the Reynolds transport theorem for the case in which $f(\mathbf{r}, t) = xyt$.

4.* A volume $V(t)$ in the form of a rectangular parallelepiped has the corners of its base at the points $(0, -\pi/2, 0)$, $(\pi, -\pi/2, 0)$, $(\pi, \pi/2, 0)$, and $(0, \pi/2, 0)$ with its sides perpendicular to the (x, y)-plane and the corners of its top surface at the points $(0, -\pi/2, z)$, $(\pi, -\pi/2, z)$, $(\pi, \pi/2, z)$, and $(0, \pi/2, z)$, where $z = ut$, with t the time and $u > 0$ a constant speed. Verify the Reynolds transport theorem for the case in which $f(\mathbf{r}, t) = \sin x \cos y e^z t^2$.

5.* A cylindrical volume $V(t)$ of height h has the center of its circular base located at the origin on the plane $z = 0$ and a radius $r = ut$, where t is the time and $u > 0$ is a constant speed. Verify the Reynolds transport theorem given that $f = r^2 t$.

6.* A hemispherical volume $V(t)$ lies in the region $z > 0$ with its center located at the origin in the plane $z = 0$ and a radius $r = ut$, where t is the time and $u > 0$ is a constant speed. Verify the Reynolds transport theorem given that $f = r^3 t$.

12.4 Fluid Mechanics Applications of Transport Theorems

When using the transport theorems, in fluid mechanics and elsewhere, two different types of time derivative occur, and for what is to follow it is important to distinguish between them. Consider a moving continuous medium, like a fluid, that has a property f associated with it, say its density, that depends on position $\mathbf{r}$ and the time t so that $f = f(\mathbf{r}, t)$. One way of finding the time derivative of f is to regard $\mathbf{r}$ as a fixed point, and then to find the time rate of change of f as seen by an observer fixed at point $\mathbf{r}$. This time derivative is denoted by $\partial f/\partial t$, and it is evaluated by differentiating f with respect to t while keeping $\mathbf{r}$ fixed. The other physically important time derivative of f involves letting the position vector $\mathbf{r}$ be a point that moves with the medium, so that $\mathbf{r} = \mathbf{r}(t)$, and then finding the time derivative of f at the moving point $\mathbf{r}$. This time derivative of f is denoted by df/dt, and in continuum mechanics it is called the **material derivative** of f, or sometimes the **convected derivative** of f, in which case it is often represented by Df/Dt.

To find the connection between the derivatives $\partial/\partial t$ and d/dt, when finding df/dt it is necessary to allow for the fact that the position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, so that $f = f(\mathbf{r}(t), t)$. Thus, allowing for the time variation in $\mathbf{r}(t)$, we

have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad \text{or} \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{q}\cdot\nabla)f,$$

where $\mathbf{q} = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}$ is the velocity of the moving point $\mathbf{r}(t)$. This shows that the material derivative operation can be written

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{q}\cdot\nabla). \tag{12}$$

Before proceeding further, notice that an application of the divergence theorem to the last term in Reynolds' transport theorem (Theorem 12.6) allows it to be written in the equivalent form

$$\frac{d}{dt}\iiint_{V(t)} f\,dV = \iiint_{V(t)} \left\{\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{q})\right\} dV, \tag{13}$$

but from Theorem 11.6 (iii) div $(f\mathbf{q}) = f(\nabla\cdot\mathbf{q}) + (\mathbf{q}\cdot\nabla)f$, so

$$\frac{d}{dt}\iiint_{V(t)} f\,dV = \iiint_{V(t)} \left\{\frac{\partial f}{\partial t} + (\mathbf{q}\cdot\nabla)f + f(\nabla\cdot\mathbf{q})\right\} dV.$$

Finally, if we use (12) this becomes

$$\frac{d}{dt}\iiint_{V(t)} f\,dV = \iiint_{V(t)} \left\{\frac{df}{dt} + f(\nabla\cdot\mathbf{q})\right\} dV. \tag{14}$$

Let us now use this result to derive the *equation of continuity* of fluid mechanics that describes the *conservation of mass* in any volume containing fluid in which fluid is not added (by a *source*) or removed (by a *sink*). To do this we assume that $V(t)$ is an arbitrary *material* volume in a fluid, so that $V(t)$ always contains the same fluid particles and the points on the surface $S(t)$ enclosing $V(t)$ move with the fluid. If we set $f = \rho$, where $\rho(\mathbf{r}, t)$ is the density of the fluid, the mass m of fluid in $V(t)$ is

$$m = \iiint_{V(t)} \rho(\mathbf{r}, t)dV.$$

As $V(t)$ is a material volume, provided it contains neither sources, nor sinks, the mass m must remain constant, from which it follows that $dm/dt = 0$.

Setting $f = \rho$ in (14), we find that

$$\frac{dm}{dt} = \iiint_{V(t)} \left\{\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{q})\right\} dV = 0.$$

As $V(t)$ is arbitrary, this is only possible if the integrand is identically zero, so that

$$\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{q}) = 0, \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0. \tag{15}$$

These are two equivalent forms of the **equation of continuity** of a fluid, which is of fundamental importance in the study of fluid dynamics.

If the fluid velocity is such that $\nabla \cdot \mathbf{q} = 0$ (div $\mathbf{q} = 0$), setting $f = 1$ in (14) reduces it to

$$\frac{d}{dt}\iiint_{V(t)} dV = \iiint_{V(t)} \nabla \cdot \mathbf{q}\, dV.$$

If div $\mathbf{q} = 0$, then $\rho_t + \rho\nabla \cdot \mathbf{q} = 0$ simplifies to $d\rho/dt = 0$. So, if initially $\rho_0 = \rho|_{t=0}$ is constant, ρ must remain constant throughout the flow even when the fluid is compressible. As $\iiint_{V(t)} dV = V$, where V is the volume of the fluid, it follows from $d/dt \iiint_{V(t)} dV = \iiint_{V(t)} \nabla \cdot \mathbf{q}\, dV$ that $dV/dt = 0$ when $\nabla \cdot \mathbf{q} = 0$. Consequently, in this case, the fluid motion will evolve without change of volume, even though the fluid may be compressible. In fluid mechanics, a flow of a compressible fluid that takes place without a change of volume is called **isochoric** flow. Naturally this last result is true when the fluid is incompressible, because then the density ρ is an absolute constant.

Next we derive a generalization of Theorem 12.6 that allows the function $f(\mathbf{r}, t)$ to be discontinuous across some surface Σ in $V(t)$ that moves with an arbitrary velocity $\mathbf{u}$, with $f = f_1(\mathbf{r}, t)$ on one side of Σ and $f = f_2(\mathbf{r}, t)$ on the other side. Particular cases of this result are needed when a physical quantity of interest experiences a discontinuous change across a surface, as can happen, for example, in chemical engineering and fluid mechanics.

The situation is illustrated in Fig. 12.15, where a material volume $V(t)$ with bounding surface $S(t)$ is shown divided into two parts $V_1(t)$ and $V_2(t)$ by a surface Σ that moves with an arbitrary velocity $\mathbf{u}$. The volume $V_1(t)$ is bounded by the surface $S_1(t)$ that is part of $S(t)$ and Σ, where the unit normal $\mathbf{n}_1$ to Σ directed out of $V_1(t)$ is $\mathbf{n}_1 = \boldsymbol{\nu}$. Similarly, volume $V_2(t)$ is bounded by the surface $S_2(t)$ that is part of $S(t)$ and Σ, where the unit normal $\mathbf{n}_2$ to Σ directed out of $V_2(t)$ is in the opposite sense to that of $\mathbf{n}_1$ so that $\mathbf{n}_2 = -\boldsymbol{\nu}$.

Applying Theorem 12.6 to volume $V_1(t)$ gives

$$\frac{d}{dt}\iiint_{V_1(t)} f_1\, dV = \iiint_{V_1(t)} \frac{\partial f_1}{\partial t} dV + \iint_{S_1(t)} f_1 \mathbf{q} \cdot d\mathbf{S} + \iint_{\Sigma(t)} f_1 \mathbf{u} \cdot \mathbf{n}_1\, dS,$$

and an application of Theorem 12.6 to the volume $V_2(t)$ gives

$$\frac{d}{dt}\iiint_{V_2(t)} f_2\, dV = \iiint_{V_2(t)} \frac{\partial f_2}{\partial t} dV + \iint_{S_2(t)} f_2 \mathbf{q} \cdot d\mathbf{S} + \iint_{\Sigma(t)} f_2 \mathbf{u} \cdot \mathbf{n}_2\, dS.$$

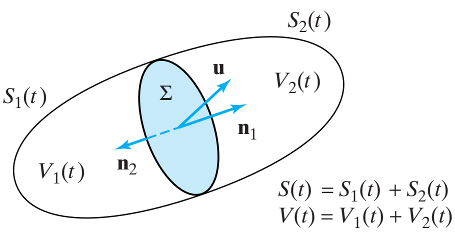


FIGURE 12.15 The material volume $V(t)$ and the surface Σ across which f is discontinuous.

Adding these two results and using the fact that $\mathbf{n}_1 = \boldsymbol{\nu}$ and $\boldsymbol{n}_2 = -\boldsymbol{\nu}$, we obtain

$$\frac{d}{dt}\iiint_{V(t)} f dV = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{S(t)} f\mathbf{q}\cdot d\mathbf{S} + \iint_{\Sigma(t)} (f_1 - f_2)\mathbf{u}\cdot d\mathbf{S}, \qquad (16)$$

which is the required generalization.

Examination of the last term in (16) shows, as would be expected, that the contribution made by the jump discontinuity $f_1 - f_2$ across the surface Σ that moves with velocity $\mathbf{u}$ depends only on the component of $\mathbf{u}$ normal to Σ, so if $\mathbf{u}$ is tangential to Σ, this term will vanish.

An extension of these ideas to allow for discontinuous solutions f in a volume $V(t)$ when f satisfies an equation of the form

$$\frac{\partial f}{\partial t} + \operatorname{div} \mathbf{h}(f) = 0,$$

called a **conservation equation**, is to be found in Chapter 18, Section 18.4, where conservation equations and shock solutions are considered. It should be noticed that an equation of this type has already been encountered in (15) when deriving the continuity equation for a fluid (the *conservation of mass equation*) in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{q}) = 0.$$

This is a *partial differential equation*, because it is an equation relating partial derivatives of the dependent variables ρ and $\mathbf{q}$.

Let Γ be a closed curve in a fluid flow with velocity vector $\mathbf{q}$ for which $\operatorname{div}\mathbf{q} = 0$ (an *isochoric flow*), and let S be any smooth surface with boundary Γ. Then the streamlines passing through Γ define a stream tube in the fluid flow. The integral

$$\Phi = \iint_S \mathbf{q}\cdot d\mathbf{S} \qquad (17)$$

is called the **strength** of the stream tube, and it measures the flow rate through the tube. As a final application of an integral theorem, we will prove that the strength of the flow in a tube bounded by streamlines (a stream tube) remains constant along its length.

First we rewrite Theorem 12.5, which was proved in the form

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q}\right]\cdot d\mathbf{S} + \int_{\Gamma(t)} \mathbf{F}\times\mathbf{q}\cdot d\mathbf{r}.$$

If we apply Stokes' theorem to the last integral, this becomes

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q} + \nabla\times(\mathbf{F}\times\mathbf{q})\right]\cdot d\mathbf{S}. \qquad (18)$$

Replacing $\mathbf{F}$ by $\mathbf{q}$, we have

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iint_{S(t)} \left[\frac{\partial \mathbf{q}}{\partial t} + (\nabla\cdot\mathbf{q})\mathbf{q} + \nabla\times(\mathbf{q}\times\mathbf{q})\right]\cdot d\mathbf{S},$$

but $\mathbf{q}\times\mathbf{q} = \mathbf{0}$, and as the flow is isochoric, $(\nabla\cdot\mathbf{q}) = 0$, this result reduces to

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iint_{S(t)} \frac{\partial \mathbf{q}}{\partial t}\cdot d\mathbf{S}.$$

An application of the divergence theorem to the integral on the right, where the closed surface $V(t)$ is formed by $S(t)$, $S(t+dt)$ and streamlines through Γ, gives

$$\frac{d}{dt}\iint_{S(t)} \mathbf{q}\cdot d\mathbf{S} = \iiint_{V(t)} \nabla\cdot(\partial\mathbf{q}/\partial t)dV = \iiint_{V(t)} \partial/\partial t(\nabla\cdot\mathbf{q})dV = 0,$$

showing that the strength $\Phi = \iint_S \mathbf{q}\cdot d\mathbf{S}$ remains constant along a stream tube.

Summary

The applications considered in this section were to fluid mechanics, and they made use of the so-called material, or convected, derivative of a function f of both position and time. The determination of this derivative was seen to involve letting a position vector move with the fluid and then finding the time derivative of f at the moving point. One result obtained by means of the transport theorems was the equation of continuity of fluid mechanics. Another result used the notion of a conservation equation to establish the invariance of the flow rate (strength) in a stream tube, the walls of which are bounded by streamlines.

EXERCISES 12.4

1. Prove the **Euler expansion formula**

$$\frac{d}{dt}\iiint_{V(t)} dV = \iint_{S(t)} \mathbf{q}\cdot d\mathbf{S}.$$

2. Show that the flux transport theorem given in (18) can also be written as

$$\frac{d}{dt}\iint_{S(t)} \mathbf{F}(\mathbf{r},t)\cdot d\mathbf{S}$$
$$= \iint_{S(t)} \left[\frac{d\mathbf{F}}{dt} + (\nabla\cdot\mathbf{q})\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{q}\right]\cdot d\mathbf{S}.$$

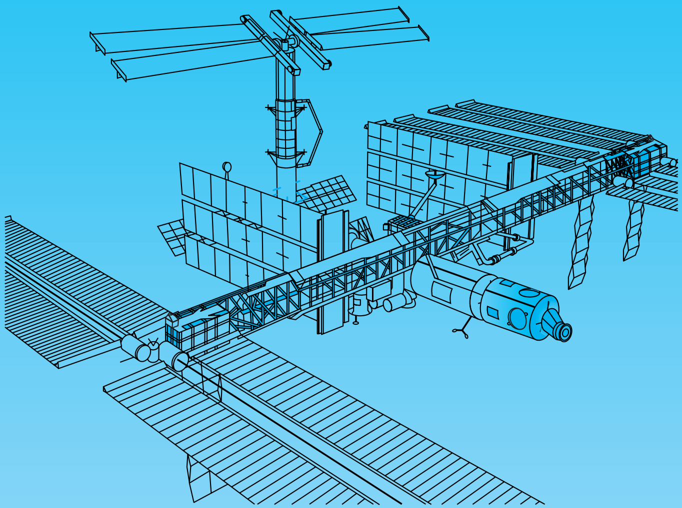
3.* Show that if

$$\frac{\partial\mathbf{F}}{\partial t} + (\nabla\cdot\mathbf{F})\mathbf{q} + \nabla\times(\mathbf{F}\times\mathbf{q}) = \mathbf{0},$$

the strength of flow through any stream tube remains constant along its length.

P A R T S I X

COMPLEX ANALYSIS



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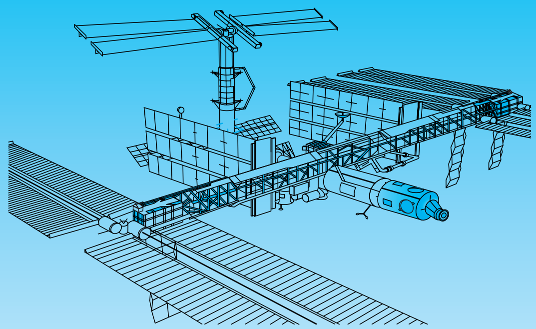
CHAPTER 13



Analytic Functions

Analytic functions involve an extension of the calculus to complex functions, and they find applications throughout all of engineering and science. Examples of direct applications are to be found in two-dimensional problems in elasticity, fluid mechanics, and electrostatics, and such functions also contribute indirectly to many other applications through their use with the Laplace and Fourier transforms. The fundamental idea underlying the systematic development of analytic functions is the extension of the concept of a derivative to a function of a complex variable. The requirement that the derivative of a complex function be independent of the way the defining complex limit is evaluated is more restrictive than the definition of partial derivatives of functions of two real variables, and it leads directly to the Cauchy–Riemann equations, which are central to the development of the subject.

After a brief review of the notion of a mapping, the fundamental concepts of the limit, continuity and differentiability of a complex function are introduced, and the essential difference between derivatives of real and complex functions is explained. An analytic function is defined, and the requirement that the limiting operation in the definition of a derivative of a complex function should be independent of the direction in which it is evaluated is shown to lead to the important Cauchy–Riemann equations. These equations provide a condition that ensures that a function of a complex variable is analytic, and both the real and imaginary parts of an analytic function are shown to be harmonic functions. Some important elementary analytic functions are defined and the problem of finding their inverse is examined.

13.1 Complex Functions and Mappings

A typical example of a complex function is the nth degree polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n, \tag{1}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers and $z = x + iy$ is an arbitrary complex variable. Assigning to z the specific value z_1 determines a complex

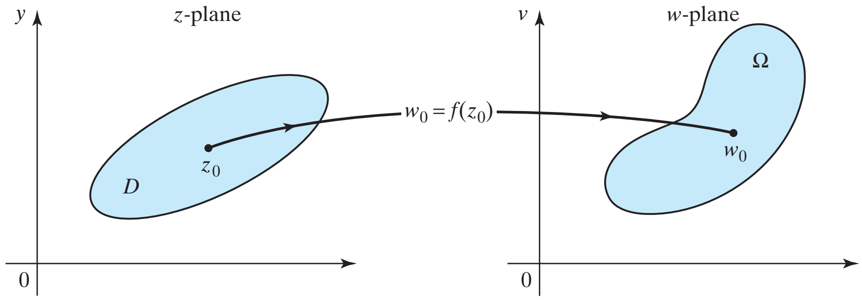


FIGURE 13.1 The function $w = f(z)$ and the w- and z-planes.

number $P(z_1)$, so to each z in the complex plane, there corresponds another complex number $P(z)$. Complex polynomials are defined for all z in the complex plane, and $P(z)$ ranges over all of the complex numbers defined by (1).

The general concept of an arbitrary complex function $w = f(z)$ can be introduced by considering two complex planes, one the z-plane containing the points $z = x + iy$ and the other the w-plane containing the points $w = u + iv$, as shown in Fig. 13.1. To develop this idea further, let a set of points D in the z-plane be such that to each point z in D there corresponds a unique complex number w belonging to another set of points Ω in the w-plane. Then the set D is said to be **mapped** onto the set Ω by a **single-valued** function of the complex variable z. A point w_0 in the w-plane corresponding to a point z_0 in the z-plane is called the **image** of z_0. The term *single-valued* is used because, by hypothesis, each point of D corresponds to one and only one point of Ω, and the name *mapping* is used because an arbitrary curve in D will correspond (be *mapped*) to a corresponding curve in Ω, with each point of the curve in Ω the image of a point in D. The notion of a mapping is important, and it will be used later in Chapter 17 when the concept of *conformal mapping* is introduced. The relationship between the points in D and the corresponding points in Ω is shown by the usual functional notation

mappings and images

$$w = f(z). \tag{2}$$

Set D is called the **domain of definition** of the complex function $f(z)$, and set Ω is called its **range**.

This definition of a function of a complex variable is more general than we require, because it places no restriction on the nature of the sets D and Ω. In complex analysis we will only be concerned with sets of points that possess the property of being *connected*. A set G will be said to be **connected** if every pair of points in G can be joined by an unbroken path with the property that every point of the path also belongs to G. Here, the path may be either a curve or a set of straight line segments joined end to end.

neighborhoods and boundaries

A **neighborhood** of a point z_0 in G is defined as all the points of a set contained strictly inside a circle of arbitrarily small radius with its center at z_0. A point z_0 is called an **interior** point of G if a neighborhood of z_0 only contains points of G. If a neighborhood of z_0 contains no points of G, the point z_0 is called an **exterior** point of G. When any neighborhood of z_0 contains both interior and exterior points of G, the point z_0 is called a **boundary point** of G. Collectively, the set of all boundary points is called the **boundary** of the set. In the sets to be considered later, the boundary

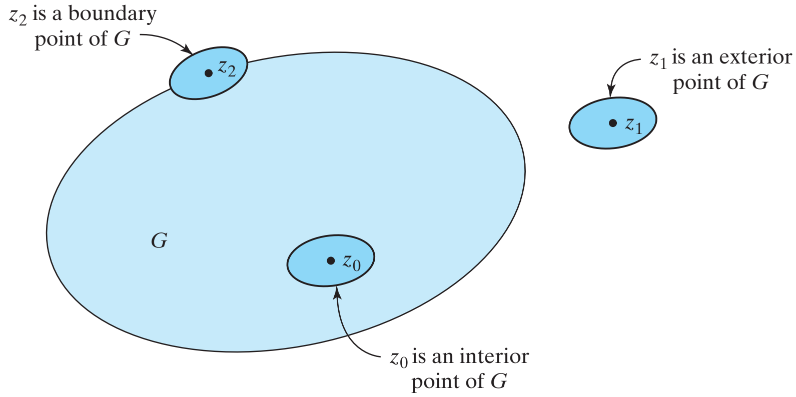


FIGURE 13.2 Interior, exterior, and boundary points of G and their associated neighborhoods.

points usually comprise a combination of straight line segments and curved arcs joined end to end to form a continuous **boundary**.

open and closed sets, and connectivity

A set G that contains no boundary points is called an **open** set. If every boundary point of set G belongs to G, then G said to be **closed**. The name **domain** is given to an open connected set, while the more general term **region** is used to describe a connected set of points that may contain none, some, or all of its boundary points. A typical open connected set G is the disc $|z| < 1$ in the z-plane. The set is *connected* because every point in G can be joined to every other point in G by a curve lying entirely inside G, and the set is *open* because however close a point in G is to the circle $|z| = 1$, a neighborhood of z_0 can always be found that only contains points of G. This becomes a closed set if the relation $|z| < 1$ is replaced by $|z| \leq 1$, because then the *boundary* of G formed by the circle $|z| = 1$ belongs to the set. These ideas are illustrated in Fig. 13.2.

In what follows we will be concerned with functions with the property that to a single element in their domain there corresponds a single element in their range and, conversely, to a single element in their range there corresponds a single element in their domain. Functions of this type are said to be **one-one**, so a function like $w = \sqrt{z}$ is to be regarded as two separate functions, each with the same domain in the z-plane, but with different ranges in the w-plane.

representing complex functions in cartesian and polar forms

The complex function (2) can be written in its **cartesian form** as

$$f(z) = u(x, y) + iv(x, y) \quad \text{for } z = x + iy \in D, \tag{3}$$

where $u(x, y)$ and $v(x, y)$ are real functions of the real variables x and y denoted by

$$u(x, y) = \mathrm{Re}\{f(z)\} \quad \text{and} \quad v(x, y) = \mathrm{Im}\{f(z)\}. \tag{4}$$

Similarly, when z is expressed in **modulus argument form** by setting $z = re^{i\theta}$, with $r = |z|$, $\theta = \mathrm{Arg}\, z$ and $-\pi < \theta \leq \pi$, the complex function $f(z)$ takes the **polar form**

$$f(z) = u(r, \theta) + iv(r, \theta), \tag{5}$$

where $u(r, \theta)$ and $v(r, \theta)$ are real functions of the real variables r and θ given by

$$u(r, \theta) = \mathrm{Re}\{f(z)\} \quad \text{and} \quad v(r, \theta) = \mathrm{Im}\{f(z)\}. \tag{6}$$

EXAMPLE 13.1

Write the function $f(z) = z^2 - z + 2$ in both its cartesian and polar form, and in each case identify the functions u and v.

Solution To arrive at the cartesian form we set $z = x + iy$ in $f(z)$ to obtain

$$\begin{aligned} f(z) = u + iv &= (x + iy)^2 - (x + iy) + 2 \\ &= x^2 + 2ixy - y^2 - x - iy + 2 \\ &= x^2 - y^2 - x + 2 + i(2xy - y). \end{aligned}$$

Equating the real and imaginary parts gives

$$u(x, y) = \mathrm{Re}\{f(z)\} = x^2 - y^2 - x + 2 \quad \text{and} \quad v(x, y) = \mathrm{Im}\{f(z)\} = 2xy - y.$$

The polar form is obtained by setting $z = re^{i\theta}$ in $f(z)$ to obtain

$$\begin{aligned} f(z) = u + iv &= r^2 e^{2i\theta} - re^{i\theta} + 2 \\ &= r^2(\cos 2\theta + i \sin 2\theta) - r(\cos\theta + i\sin\theta) + 2 \\ &= r^2 \cos 2\theta - r\cos\theta + 2 + i(r^2 \sin 2\theta - r\sin\theta). \end{aligned}$$

In this case, equating real and imaginary parts gives

$$u(r, \theta) = r^2 \cos 2\theta - r\cos\theta + 2 \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta - r\sin\theta.$$ ■

EXAMPLE 13.2

Draw the straight line segment in the z-plane joining the points $z = 2 + 3i$ and $z = 4 + 5i$, and find its image in the w-plane under the mapping $w = \frac{1}{2}z + i$.

Solution The straight line segment starts at the point A with coordinates $(2, 3)$ and ends at the point B with coordinates $(4, 5)$, so if it has the equation $y = mx + c$, its gradient $m = (5 - 3)/(4 - 2) = 1$. As the line must pass through the point $(2, 3)$, substitution into the equation $y = mx + c$ gives $3 = 2 + c$, so $c = 1$. This has established that the equation of the line to which the line segment AB belongs is $y = x + 1$.

The mapping is $w = \frac{1}{2}z + i$, so setting $w = u + iv$ and $z = x + iy$, we find that $u + iv = \frac{1}{2}x + i(\frac{1}{2}y + 1)$. Equating the real and imaginary parts of this equation gives $u = \frac{1}{2}x$, $v = \frac{1}{2}y + 1$. As the straight line segment AB in the z-plane is part of the line $y = x + 1$, substituting for x and y in terms of u and v shows that the mapping onto the w-plane of the line to which AB belongs has the equation $v = u + 3/2$. This is also the equation of a straight line, so we have established that $w = \frac{1}{2}z + i$ maps the straight line $y = x + 1$ in the z-plane onto the straight line $v = u + 3/2$ in the w-plane.

To draw the required image in the w-plane we must now determine the images A' and B' in the w-plane of A and B in the z-plane, and then join them by a straight line. As A is the point $z = 2 + 3i$ and B is the point $z = 4 + 5i$, substitution into $w = \frac{1}{2}z + i$ shows that A' is the point $w = 1 + \frac{5}{2}i$ and B' is the point $w = 2 + \frac{7}{2}i$. The line segments in the z- and w-planes are shown in Fig. 13.3. ■

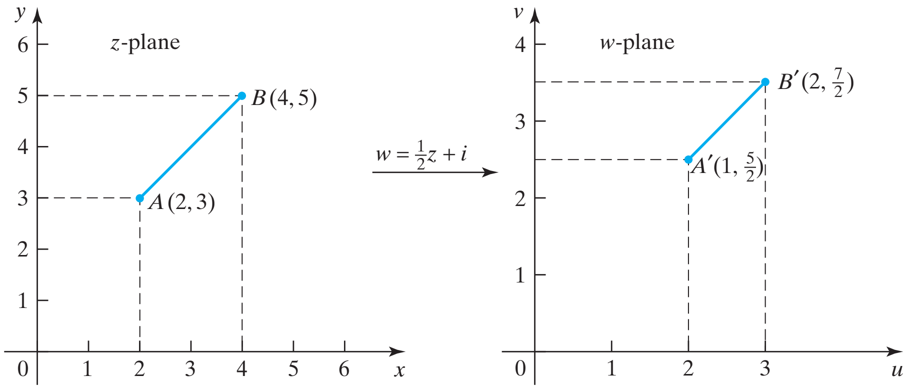


FIGURE 13.3 The image of line AB under the mapping $w = \frac{1}{2}z + i$.

EXAMPLE 13.3

(a) Draw and shade the area in the z-plane containing the points satisfying the conditions $|z - 1 + 2i| \le 1$ and $\text{Im}\{z\} > -2$, marking a boundary that belongs to the set by a solid line and one that does not belong to it by a dashed line. (b) Draw and shade the area in the z-plane to which belong the points satisfying the conditions $r = |z - 1| \ge 2$ and $\pi/6 \le \text{Arg}(z - 1) \le \pi/3$.

Solution

(a) We must use the fact that the modulus of a complex number is a nonnegative real number, and $|z_1 - z_2|$ is the distance between z_1 and z_2. It follows from this that the inequality $|z - 1 + 2i| \le 1$ is satisfied by all points z distant from the point $1 - 2i$ by an amount less than or equal to 1. So the inequality $|z - 1 + 2i| \le 1$ is satisfied by all points inside and on a circle of radius 1 centered on the point $1 - 2i$. As $\text{Im}\{z\} = y$, the inequality $\text{Im}\{z\} > -2$ is simply $y > -2$. So the required points lie inside and on a circle of radius 1 centered on the point $1 - 2i$, and strictly above the line $y = -2$. The required area is shown in Fig. 13.4a, where the boundary of the circle has been drawn using a solid line because these boundary points belong

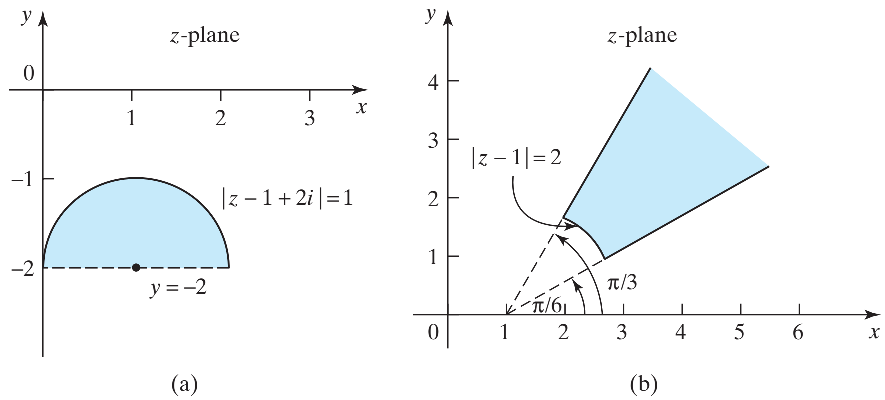


FIGURE 13.4 (a) Points satisfying $|z - 1 + 2i| \le 1$, $\text{Im}\{z\} > -2$. (b) Points satisfying $r = |z - 1| \ge 2$ and $\pi/6 \le \text{Arg}(z - 1) \le \pi/3$.

to the set, while the bounding line $y = -3$ is drawn as a dashed line because points on this boundary do not belong to the set.

(b) The condition $r = |z - 1| \geq 2$ is satisfied by all points outside and on a circle of radius 2 with its center at $z = 1$, and a condition of the form $\mathrm{Arg}(z - 1) = \omega$ is a radial line drawn from the point $z = 1$ as origin making an angle ω measured counterclockwise from the positive real axis. Thus the condition $\pi/6 \leq \mathrm{Arg}(z - 1) \leq \pi/3$ gives a wedge shaped area in the upper half of the z-plane centered on the point $z = 1$ with its bounding lines making angles $\pi/6$ and $\pi/3$ with the positive real axis. The required area is shown in Fig. 13.4b. ■

EXAMPLE 13.4 Find the image of the set of points $|\mathrm{Re}\{z\}| < 1$, $|\mathrm{Im}\{z\}| < 2$ in the z-plane under the mapping $w = 2z + 1$.

Solution When mapping areas, the approach to be used is first to determine how the boundary transforms, and then to determine if the points in the given area in the z-plane map to points inside or outside the image of this boundary in the w-plane. As $\mathrm{Re}\{z\} = x$ and $\mathrm{Im}\{z\} = y$, the area in the z-plane lies inside the rectangle $-1 < x < 1$, $-2 < y < 2$ shown in the left of Fig. 13.5.

Setting $z = x + iy$ in $w = 2z + 1$ gives $w = u + iv = 2x + 1 + 2iy$, so $u = 2x + 1$ and $v = 2y$. The top boundary of the area in the z-plane in Fig. 13.5 is $-1 < x < 1$, $y = 2$, so using these results in the mapping shows the image of this boundary in the w-plane to be given by $u = 2x + 1$, with $-1 < x < 1$, and $v = 4$. A repetition of this form of argument applied to the other three sides of the rectangle establishes that the image in the w-plane of the rectangle in the z-plane is the one illustrated on the right of Fig. 13.5. A general point (x, y) inside the rectangle in the z- plane maps to the point $(2x + 1, 2y)$ in the w-plane with $-1 < x < 1$, $-2 < y < 2$, and this point is seen to lie inside the rectangular boundary in the w-plane. Consequently, all points inside the rectangle in the z-plane map to points inside the image rectangle in the w-plane. Inspection of Fig. 13.5 shows that the geometrical effect of this mapping is first to scale the rectangle in the z-plane uniformly by a factor 2 in both the x and y directions, and then to shift the origin parallel to the real axis. Mappings are examined in greater detail in Chapter 17 in connection with conformal mappings. ■

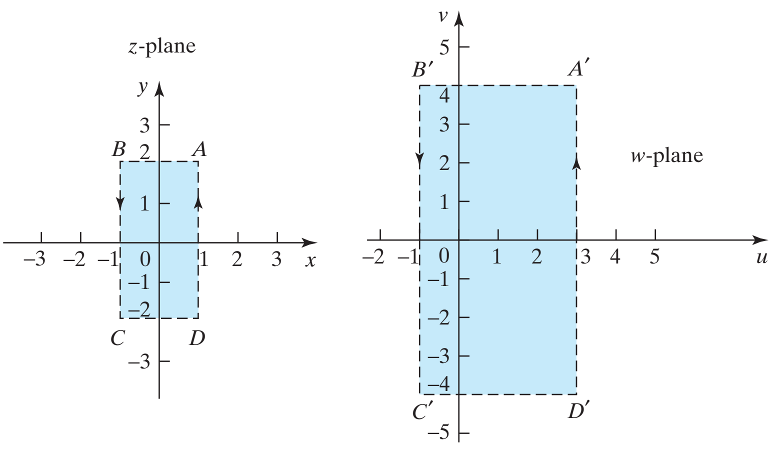


FIGURE 13.5 The effect of the mapping $w = 2z + 1$ on a rectangle.

Summary A mapping by a single-valued complex function and the image of a point were defined, the notion of a connected set was introduced, and the definition of a neighborhood was used to define the boundary of a set in the complex plane and to identify open and closed regions in the complex plane.

EXERCISES 13.1

In Exercises 1 and 2 sketch and shade the areas in the z-plane occupied by points satisfying the given conditions. Represent a boundary that belongs to a set by a solid line and one that does not by a dashed line. Determine if the areas represent open sets, closed sets, or regions.

1. (a) $|z| \geq 1$ and $|z| \leq 2$. (b) $|z - i| \leq 1$ and $|z| < 1$. (c) $0 < x < 1,\ 0 < y < 1$.

2. (a) $1 < |z| \leq 2$. (b) $1 < |z - 1| \leq 2, x > 1, y > 0$. (c) $\mathrm{Re}\{z\} > 0, \mathrm{Im}\{z\} < 0, |z| \leq 2$.

3. Determine the image of the straight line segment joining the origin to the point $z = 2 + 2i$ under the mapping $w = -iz$.

4. Set $w = u + iv$ and use the fact that $z\bar{z} = 1$ on the circle $|z| = 1$ to determine the image under the mapping $w = 2z - 1$ of the part of the circular arc $|z| = 1$ that lies in the first quadrant of the z-plane.

5. Determine the image of the points satisfying $|\mathrm{Re}\{z\}| > 2, |\mathrm{Im}\{z\}| < 1$ in the z-plane under the mapping $w = iz + 2$.

6. Determine the image of the points satisfying $|\mathrm{Re}\{z\}| > 4, |\mathrm{Im}\{z\}| > 2$ in the z-plane under the mapping $w = i - 3z$.

7. By considering the lines joining the origin and the point (2, 0) to a point z in the upper half of the z-plane, show that the conditions $\mathrm{Arg}(z - 2) - \mathrm{Arg}\, z = \pi/2$ and $0 \leq \mathrm{Arg}\, z \leq \pi/2$ define a semicircular arc of radius 1 in the upper half of the z-plane with its center at $z = 1$.

8. By considering the lines joining the points (1, 0) and (3, 0) to a point z in the upper half of the z-plane, determine the area in the z-plane defined by the conditions $\mathrm{Arg}(z - 3) - \mathrm{Arg}(z - 1) = \pi/2$, $0 \leq |z - 2| \leq 1$, and $\pi/4 \leq \mathrm{Arg}(z - 2) \leq 3\pi/4$.

9. Use a geometrical argument to find the locus of points z such that

$$|z - 1| + |z + 1| = 4.$$

10. Use a geometrical argument to find the locus of points z such that

$$|z - 3i| = |z - i|.$$

Express the functions in Exercises 11 through 14 in both cartesian and polar form, and determine the forms taken by u and v in each case.

11. $f(z) = (2z + i)/(z + i)$.

12. $f(z) = 3z^2 - 2z + 1/z$.

13. $f(z) = ze^{iz}$.

14. $f(z) = z + 1/z$.

13.2 Limits, Derivatives, and Analytic Functions

When working with functions of a complex variable it is necessary to generalize the related concepts of a limit and continuity by extending the corresponding definitions from real analysis. These generalizations use the fact that in the complex plane the modulus $|z|$ measures the magnitude of z, so $|z_1 - z_2|$ can be considered to measure the distance between points z_1 and z_2 in the z-plane. The function

$$f(z) = u(x, y) + iv(x, y) \tag{7}$$

complex limit

will have the **complex limit** L, written

$$\lim_{z \to z_0} f(z) = L = L_1 + iL_2, \tag{8}$$

where L_1 and L_2 are real numbers, if

$$\lim_{|z-z_0|\to 0} |f(z) - L| = 0. \tag{9}$$

If $z = x + iy$ and $z_0 = x_0 + iy_0$, then z will tend to z_0, written $z \to z_0$, when $(x, y) \to (x_0, y_0)$, so (9) is equivalent to

$$\lim_{(x,y)\to(x_0,y_0)} |f(z) - L| = 0. \tag{10}$$

However, by the triangle inequality,

$$\begin{aligned}|f(z) - L| &= |u(x, y) + iv(x, y) - L_1 - iL_2| = |u(x, y) - L_1 + i(v(x, y) - L_2)|\\ &\le |u(x, y) - L_1| + |v(x, y) - L_2|,\end{aligned}$$

so in terms of real functions, $f(z)$ will have the limit L as $z \to z_0$ if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = L_2. \tag{11}$$

This shows the connection between the limit of a function $f(z)$ of a complex variable and the limits of the real functions $u(x, y)$ and $v(x, y)$. Because of this relationship, the fundamental properties of limits of functions of a real variable are transferred to functions of a complex variable, with the result that if $f(z)$ and $g(z)$ have limits as $z \to z_0$, then

$$\lim_{z\to z_0} [f(z) \pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z) \tag{12}$$

$$\lim_{z\to z_0} [f(z)g(z)] = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z) \tag{13}$$

$$\lim_{z\to z_0} [f(z)/g(z)] = \lim_{z\to z_0} f(z)/ \lim_{z\to z_0} g(z), \quad \text{when} \quad \lim_{z\to z_0} g(z) \neq 0. \tag{14}$$

continuous and discontinuous complex functions

As with real functions of a real variable, the complex function $f(z)$ will be said to be **continuous** at z_0 if it is defined in a neighborhood of z_0 and $f(z_0)$ exists and is equal to $\lim_{z\to z_0} f(z)$. When expressed in terms of real functions, it can be seen that $f(z) = u + iv$ will be continuous at $z_0 = x_0 + iy_0$ if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v(x_0, y_0). \tag{15}$$

A function $f(z)$ that does not satisfy condition (15) at (x_0, y_0), that is, at $z = z_0$, will be said to be **discontinuous** at z_0.

It is a direct consequence of the definitions of a limit and of continuity that the sum and difference of continuous complex functions of a complex variable are themselves continuous, and the quotient of continuous functions is continuous at z_0 provided the divisor does not vanish at z_0.

EXAMPLE 13.5

Examine the continuity of the functions (a) $f(z) = z^2 + 3z - 1$ and (b) $f(z) = z/(z-1)$.

Solution

(a) Setting $z = x + iy$ in $f(z)$ and identifying the real and imaginary parts gives

$$\begin{aligned} f(z) &= (x+iy)^2 + 3(x+iy) - 1 \\ &= x^2 - y^2 + 3x - 1 + i(2xy + 3y), \end{aligned}$$

so if $f(z) = u + iv$, then

$$u(x, y) = x^2 - y^2 + 3x - 1 \quad \text{and} \quad v(x, y) = 2xy + 3y.$$

As u and v are continuous for all (x, y), that is, for all z, it follows from (15) that $f(z)$ is continuous for all z.

(b) The function $f(z)$ can be considered as the product of the functions $g(z) = z$ and $h(z) = 1/(z-1)$, and clearly $g(z)$ is continuous for all z. To examine the behavior of $h(z)$ we set $z = x + iy$, and after separating the real and imaginary parts we have

$$h(z) = \frac{1}{x + iy - 1} = \frac{x-1}{(x-1)^2 + y^2} - i\frac{y}{(x-1)^2 + y^2}.$$

So, if $h(z) = u_2 + v_2$, then

$$u_2(x, y) = \frac{x-1}{(x-1)^2 + y^2} \quad \text{and} \quad v_2(x, y) = -\frac{y}{(x-1)^2 + y^2}.$$

The functions u_2 and v_2 are continuous for all (x, y) except at the point $(1, 0)$ corresponding to $z = 1$ where their divisors vanish. Thus, $h(z)$ is continuous for all z except at $z = 1$, so it follows from (13) that the product $f(z) = g(z)h(z)$ is continuous everywhere except at the point $z = 1$, where it has a discontinuity.

This same conclusion can be reached if $f(z)$ is regarded as a quotient of the functions $g(z) = z$ and $h(z) = (z-1)$. Setting $z = x + iy$ in $f(z)$ and identifying the real and imaginary parts gives

$$f(z) = \frac{x + iy}{x + iy - 1} = \frac{x^2 + y^2 - x}{(x-1)^2 + y^2} - i\frac{y}{(x-1)^2 + y^2},$$

so if $f(z) = u + iv$, then

$$u(x, y) = \frac{x^2 + y^2 - x}{(x-1)^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{(x-1)^2 + y^2}.$$

Both u and v have limits as $(x, y) \to (x_0, y_0)$ for all points (x_0, y_0) with the exception of the point $(1, 0)$, corresponding to $z = 1$, where their divisors vanish. So again we conclude that $f(z)$ is continuous for all z with the exception of the point $z = 1$, where it is discontinuous. ■

derivative of a complex function

A major difference between a real-valued function of two real variables and a single-valued function of a complex variable $w = f(z) = u(x, y) + iv(x, y)$ arises when the derivative of $f(z)$ is introduced. If a single-valued complex function $f(z)$ is defined in some domain D of the complex plane then, when it exists, its **derivative**

$f'(z)$ is defined as

$$f'(z) = \frac{dw}{dz} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}, \tag{16}$$

where in the limit on the right the complex variable h is allowed to tend to zero along any path in the z-plane. It is this last condition that distinguishes the derivative of a complex function from that of a real function of two real variables because, as will be seen later, the existence of a unique derivative $f'(z)$ requires a special relationship to exist between the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(z)$.

analytic and entire functions

A function that has a continuous derivative throughout some domain D of the complex plane is said to be **analytic** in D. A function is **analytic at a point** P if there is a region containing P in which it is analytic, and a function that is analytic everywhere in the z-plane is called an **entire** function.

On account of the definition of a derivative in (16), and results (12) to (14) involving limits, it follows that the rules for the differentiation of real functions of a real variable carry over to complex functions, so for functions $f(z)$ and $g(z)$ that are analytic in D,

fundamental rules for differentiating combinations of complex functions

$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z) \quad \text{is analytic in } D \tag{17}$$

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \quad \text{is analytic in } D \tag{18}$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2} \quad \text{is analytic in } D \text{ wherever } g(z) \neq 0, \tag{19}$$

and differentiation of a composite function (*function of a function*) is given by the familiar result

$$\frac{d[f(g(z))]}{dz} = g'(z)f'(g(z)), \tag{20}$$

where the expression on the right is analytic whenever the range of $g(z)$ lies within the domain of definition of $f(z)$, and $f'(g(z))$ exists.

Higher derivatives are defined in the usual manner, so that, for example,

$$\frac{d^2[f(z)]}{dz^2} = \frac{d}{dz}\left[\frac{d[f(z)]}{dz}\right] = f''(z) \quad \text{and}$$
$$\frac{d^3[f(z)]}{dz^3} = \frac{d}{dz}\left[\frac{d^2[f(z)]}{dz^2}\right] = \frac{d[f''(z)]}{dz} = f'''(z). \tag{21}$$

It follows directly that if $f(z)$ and $g(z)$ are analytic in a common domain D of the complex plane, then $f(z) \pm g(z)$ and $f(z)g(z)$ are analytic in D, and $f(z)/g(z)$ is analytic in D except for points where $g(z) = 0$ but $f(z) \neq 0$.

The formal definition of a derivative in (16) does not usually provide a convenient way of calculating $f'(z)$, though it can be used as shown by the next example

EXAMPLE 13.6

finding an important derivative from first principles

Use the definition of a derivative in (16) to show that

$$\frac{d[z^n]}{dz} = nz^{n-1} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots,$$

and that z^n is analytic for all z when $n = 0, 1, 2, \ldots,$ and when $n = -1, -2, \ldots,$ it is analytic everywhere except at $z = 0$.

Solution We consider the cases $n = 0, 1, 2, \ldots,$ and $n = -1, -2, \ldots,$ separately.

Case: n = 0. From (16) we have

$$\frac{d[1]}{dz} = \lim_{h \to 0}\left(\frac{1-1}{h}\right) = 0,$$

and this is true irrespective of how $h \to 0$, so the statement is true for $n = 0$.

Case: n a positive integer. From (16), after expanding $(z+h)^n$ by the binomial theorem, we have

$$\begin{aligned}
\frac{d[z^n]}{dz} &= \lim_{h \to 0}\left(\frac{(z+h)^n - z^n}{h}\right) \\
&= \lim_{h \to 0}\left(\frac{z^n + nhz^{n-1} + \frac{n(n-1)}{2!}h^2 z^{n-2} + \cdots + h^n - z^n}{h}\right) \\
&= nz^{n-1} + \lim_{h \to 0} h\left(\frac{n(n-1)}{2!}z^{n-2} + \frac{n(n-1)(n-2)}{3!}hz^{n-3} + \cdots + h^{n-2}\right) \\
&= nz^{n-1}.
\end{aligned}$$

This result is also true for all z, irrespective of the path in the z-plane by which $h \to 0$, so the statement is true for all positive integers n.

Case: n a negative integer. In this case, using (19) with $f(z) = 1$ and $g(z) = z^n$ gives

$$\frac{d[z^{-n}]}{dz} = -\frac{d[z^n]/dz}{z^{2n}} = -nz^{-(n+1)}, \quad \text{for } z \neq 0,$$

so the statement in the problem is seen to be true when n is a negative integer and $z \neq 0$. We have shown that when $n = 0, 1, 2, \ldots$ the function $f(z) = z^n$ is analytic for all z, and when n is a negative integer it is analytic everywhere except at the origin. ■

The definition of a derivative in (16) is too cumbersome to use for general purposes. A more convenient way of determining derivatives will be found as a result of arriving at conditions to be satisfied by $u(x, y)$ and $v(x, y)$ that will ensure that the function $f(z) = u(x, y) + iv(x, y)$ is analytic.

THEOREM 13.1

a fundamental condition to be satisfied if a complex function is to have a derivative

Cauchy–Riemann equations The single-valued complex function

$$f(z) = u(x, y) + iv(x, y)$$

defined for all z in some domain D of the complex plane will have a derivative $f'(z)$ at every point of D, and so be analytic in D, if the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ are continuous throughout D and satisfy the **Cauchy–Riemann** equations at every point of D:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof To arrive at conditions to be satisfied by $f(z) = u + iv$ that will ensure that $f'(z)$ exists and is unique in D, independently of the way in which $h \to 0$ in (16), we will compute $f'(z)$ in two different ways. First we will find $f'(z)$ by letting $h \to 0$ parallel to the real axis, and then by letting $h \to 0$ parallel to the imaginary axis, as a result of which two different expressions will be obtained for $f'(z)$. If these are to be identical, their respective real and imaginary parts must be equal, and it will be this requirement that will lead to the Cauchy–Riemann equations.

First we set $h = h_1 + i0$ and let $h_1 \to 0$, so that $h \to 0$ parallel to the real axis, and as a result (16) becomes

$$\begin{aligned} f'(z) &= \lim_{h_1 \to 0} \left[\frac{u(x + h_1, y) + iv(x + h_1, y) - u(x, y) - iv(x, y)}{h_1} \right] \\ &= \lim_{h_1 \to 0} \left[\frac{u(x + h_1, y) - u(x, y)}{h_1} \right] + i \lim_{h_1 \to 0} \left[\frac{v(x + h_1, y) - v(x, y)}{h_1} \right] \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \end{aligned}$$

Next we set $h = 0 + ih_2$ and let $h_2 \to 0$, so that $h \to 0$ is parallel to the imaginary axis. In this case (16) becomes

$$\begin{aligned} f'(z) &= \lim_{h_2 \to 0} \left[\frac{u(x, y + h_2) + iv(x, y + h_2) - u(x, y) - iv(x, y)}{ih_2} \right] \\ &= \lim_{h_2 \to 0} \left[\frac{u(x, y + h_2) - u(x, y)}{ih_2} \right] + i \lim_{h_2 \to 0} \left[\frac{v(x, y + h_2) - v(x, y)}{ih_2} \right] \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \end{aligned}$$

Equating these two different expressions for $f'(z)$, whose respective real and imaginary parts must be equal, gives the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

that must hold throughout D if $f(z)$ is to be analytic in D.

It is somewhat harder to prove that when $u(x, y)$ and $v(x, y)$ have continuous partial derivatives u_x, u_y, v_x, and v_y in D, the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D, so the details of the proof will be omitted. ■

Augustin-Louis Cauchy (1789–1857)
A French mathematician who was born in Paris and studied and held a professorship at the Ecole Polytechnique. He was subsequently appointed to the chair of mathematical physics at the University of Turin. Cauchy published many mathematical papers, and he was responsible for introducing a rigorous definition of a limit. One of his most important contributions was to the development of complex analysis. Among his other works of a fundamental nature were contributions to number theory, differential equations, and various aspects of mathematical physics.

George Friedrich Bernhard Riemann (1826–1866)
A German mathematician of outstanding ability who was born in Hanover, but whose delicate health due to tuberculosis resulted in his untimely death while visiting Italy. He studied under Gauss, and after a period of time in Berlin he returned to Göttingen to study physics under Weber. He was made Professor of Mathematics in Göttingen in 1859, and he made contributions of fundamental importance to many branches of mathematics, some of which were influenced by his earlier studies in physics. Among his remarkable contributions, it was his work that led to a proper understanding of definite integrals and to the development of complex analysis and its geometrical interpretation.

The implications of the Cauchy–Riemann equations are far-reaching, because it will be shown later that if a function is analytic in D, then it possesses derivatives of *all* orders.

When $f(z) = u(x, y) + iv(x, y)$ is an analytic function in D, a convenient method for the computation of $f'(z)$ follows from the first expression found in Theorem 13.1, because then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{22}$$

This result expresses the derivative $f'(z)$ in its cartesian form involving functions of x and y, but it is often necessary to represent $f'(z)$ as a function of z. In general, to convert the cartesian form of an analytic function $g(z) = u(x, y) + iv(x, y)$ into an expression in terms of z, it is only necessary to recognize that when z is purely real the functional forms of $g(x)$ and $g(z)$ are identical. This leads to the following general rule.

how to convert an analytic function in (x, y) form to a function of z

Rule for converting an analytic function $w = u + iv$ to the form $w = f(z)$

Let $g(z) = u(x, y) + iv(x, y)$ be an analytic function in some domain D of the complex plane. Then the cartesian represention of the function involving x and y on the right of $g(z)$ can be converted to a function of z by setting $y = 0$ and replacing x by z in $u(x, y)$ and $v(x, y)$.

EXAMPLE 13.7

Show that $f(z) = z^2$ satisfies the Cauchy–Riemann equations and is an entire function. Use result (22) and the foregoing rule to show that

$$\frac{d}{dz}[z^2] = 2z.$$

Solution If we set $f(z) = z^2 = u + iv$, it follows that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Then $\partial u/\partial x = 2x$, $\partial u/\partial y = -2y$, $\partial v/\partial x = 2y$, and $\partial v/\partial y = 2x$, so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

showing that the Cauchy–Riemann equations are satisfied for all (x, y), so z^2 is an entire function. From (22) the cartesian form of $f'(z)$ is

$$\frac{d}{dz}[z^2] = 2x + i2y,$$

so setting $y = 0$ and replacing x by z the above rule shows that

$$\frac{d}{dz}[z^2] = 2z,$$

in agreement with the result of Example 13.6 with $n = 2$. ■

Not every function of a complex variable is an analytic function, as can be seen from the next example.

EXAMPLE 13.8

Show that neither $f(z) = \bar{z}$ nor $f(z) = |z|$ is an analytic function.

Solution Setting $f(z) = \bar{z} = x - iy$, we have $u(x, y) = x$ and $v(x, y) = -y$, so $\partial u/\partial x = 1$ and $\partial v/\partial y = -1$. As the first Cauchy–Riemann equation is not satisfied at any point in the z-plane, the function $f(z) = \bar{z}$ is not an analytic function.

Setting $f(z) = |z| = (x^2 + y^2)^{1/2}$, we find that $u(x, y) = (x^2 + y^2)^{1/2}$ and $v(x, y) \equiv 0$. As $\partial v/\partial x = \partial v/\partial y \equiv 0$, the Cauchy–Riemann equations cannot be satisfied in the z-plane, so $f(z) = |z|$ is not an analytic function. This is not surprising, because $|z|$ is a *real* function. ■

It should be recognized that because polynomials are sums of analytic functions, they are themselves analytic functions. As a result, derivatives of sums and products of polynomials are analytic functions, and derivatives of quotients of polynomials are analytic functions except at the zeros of their divisors. Derivatives of polynomials are obtained by repeated use of the result of Example 13.6 using the appropriate values of n.

EXAMPLE 13.9

Find $F'(z)$ given that $F(z) = z/(z^2 - 1)$.

Solution Applying (19) with $f(z) = z$ and $g(z) = z^2 - 1$ gives

$$\frac{d}{dz}\left[\frac{z}{z^2 - 1}\right] = -\frac{(z^2 + 1)}{(z^2 - 1)^2}, \quad \text{for } z \neq \pm 1.$$ ■

the complex exponential

It is natural to define the **complex exponential function** e^z as

$$f(z) = e^z = e^{(x+iy)} = e^x(\cos y + i \sin y), \tag{23}$$

because when $z = x + i0$ this reduces to the definition of e^x, and when $z = 0 + iy$ it becomes the Euler formula

$$e^{iy} = \cos y + i \sin y.$$

Expression (23) is compatible with the series representation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{24}$$

because when $z = x$ this becomes the ordinary exponential series for e^x with an infinite radius of convergence, and when $z = iy$ it becomes the Euler formula.

The form of argument used in elementary calculus to establish the ratio test for the convergence of a series in the real variable x remains true when x is replaced by the complex variable z and the absolute value of x is replaced by the modulus of z (see Section 15.1). As a result, because e^x has an infinite radius of convergence and so can be differentiated term by term, so can e^z, because it converges in a *disc* of arbitrarily large radius centered on the origin in the z-plane. Term-by-term differentiation of series (24) is permissible and shows that

$$\frac{d[e^z]}{dz} = e^z.$$

Replacing z in the series by az, with a an arbitrary complex constant, and again differentiating term by term gives the more general result

$$\frac{d[e^{az}]}{dz} = ae^{az}, \tag{25}$$

and so e^{az} is an entire function.

complex hyperbolic functions

As with the real variable case, the **complex hyperbolic functions** $\sinh z$ and $\cosh z$ are defined by the formulas

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}, \tag{26}$$

and after squaring and differencing these definitions we obtain the fundamental identity

$$\cosh^2 z - \sinh^2 z = 1. \tag{27}$$

Differentiation of definitions (26) with z replaced by az shows that

$$\frac{d[\sinh az]}{dz} = a\cosh z \quad \text{and} \quad \frac{d[\cosh az]}{dz} = \sinh az, \tag{28}$$

but as e^{az} is an entire function, so also are $\sinh az$ and $\cosh az$.

By definition,

$$\tanh z = \frac{\sinh z}{\cosh z}, \tag{29}$$

so after z is replaced by az, an application of (19) together with results (27) and (28) shows that

$$\frac{d[\tanh az]}{dz} = \frac{a\cosh^2 az - a\sinh^2 az}{\cosh^2 az} = \frac{a}{\cosh^2 az} = a\,\mathrm{sech}^2 az, \tag{30}$$

provided $\cosh az \neq 0$. This last condition is necessary because although the real-variable hyperbolic cosine function never vanishes, the complex hyperbolic cosine function has an infinity of zeros. The complex function $\tanh az$ is seen to be analytic in any domain D that does not contain a zero of $\cosh az$, so it is *not* an entire function.

The functions $\operatorname{sech} az$, $\operatorname{csch} az$, and $\coth az$ are defined in the usual manner as

$$\operatorname{sech} az = \frac{1}{\cosh az}, \quad \operatorname{csch} az = \frac{1}{\sinh az}, \quad \text{and} \quad \coth az = \frac{1}{\tanh az}, \tag{31}$$

with the derivatives

$$\begin{aligned}
&\frac{d}{dz}[\operatorname{sech} az] = -a \operatorname{sech} az \tanh az \text{ for } az \neq \left(n + \frac{1}{2}\right)\pi i,\\
&\frac{d}{dz}[\operatorname{csch} az] = -a \operatorname{csch} az \coth az \text{ for } az \neq n\pi i,\\
&\frac{d}{dz}[\coth az] = -a \operatorname{csch}^2 az \text{ for } az \neq n\pi i.
\end{aligned} \tag{32}$$

EXAMPLE 13.10 Find the zeros of (a) $\cosh z$ and (b) $\cos z - 3$.

Solution

(a) By definition

$$\begin{aligned}
\cosh z &= \frac{1}{2}[e^{x+iy} + e^{-x-iy}] = \frac{1}{2}e^x x^{iy} + \frac{1}{2}e^{-x}e^{-iy}\\
&= \frac{1}{2}e^x(\cos y + i \sin y) + \frac{1}{2}e^{-x}(\cos y - i \sin y)\\
&= \left(\frac{e^x + e^{-x}}{2}\right)\cos y + i\left(\frac{e^x - e^{-x}}{2}\right)\sin y\\
&= \cosh x \cos y + i \sinh x \sin y.
\end{aligned}$$

The function $\cosh z$ will vanish when

$$\begin{aligned}
u(x, y) &= \operatorname{Re}\{\cosh z\} = \cosh x \cos y = 0 \quad \text{and}\\
v(x, y) &= \operatorname{Im}\{\cosh z\} = \sinh x \sin y = 0,
\end{aligned}$$

and this is only possible if $\cos y = 0$ and $\sinh x = 0$. The function $\cos y = 0$ when $y = (2n+1)\pi/2$ for $n = 0, \pm 1, \pm 2, \ldots$, and $\sinh x = 0$ only when $x = 0$, so the *zeros* of $\cosh z$, that is, the *roots* of $\cosh z = 0$, are $z = i(2n+1)\pi/2$ for $n = 0, \pm 1, \pm 2, \ldots$.

(b) A similar argument shows that $\cos z = \cos x \cosh y - i \sin x \sinh y$, so $\cos z = 3$ if $\cos x \cosh y = 3$ and $\sin x \sinh y = 0$. The first condition is true if $\cos x = 1$ and $\cosh y = 3$, from which it follows that $y = \pm \operatorname{arccosh} 3$ (remember that the inverse hyperbolic cosine function is double valued) and $x = 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$. This choice of x also causes the second condition to be satisfied for all y, so the *zeros* of $\cos z - 3$, that is, the *roots* of $\cos z = 3$, are $z = 2n\pi \pm i \operatorname{arccosh} 3$, for $n = 0, \pm 1, \pm 2, \ldots$. ■

EXAMPLE 13.11

Use the Cauchy–Riemann equations to show that $\cosh z$ is an entire function, and to find $d[\cosh z]/dz$.

Solution It was shown in Example 13.10 that if $\cosh z = u(x, y) + iv(x, y)$, then

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Routine differentiation shows u and v satisfy the Cauchy–Riemann equations for all z, so $\cosh z$ is an entire function. Substituting in (22) gives

$$\frac{d}{dz}[\cosh z] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sinh x \cos y + i \cosh x \sin y,$$

so as $\cosh z$ is an analytic function, setting $y = 0$ and replacing x by z to express the result in terms of z, we obtain the expected result

$$\frac{d[\cosh z]}{dz} = \sinh z.$$

complex trigonometric functions

To make the complex trigonometric sine and cosine functions compatible with the definitions of the corresponding real variable trigonometric functions, we use the definitions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{33}$$

so that, in particular, when $z = x$ is real,

$$\sin ix = i \sinh x, \quad \cos ix = \cosh x, \quad \sinh ix = i \sin x, \quad \text{and} \quad \cosh ix = \cos x. \tag{34}$$

By squaring and adding the expressions in (33), we obtain the fundamental identity

$$\sin^2 z + \cos^2 z = 1. \tag{35}$$

Replacing z by az and differentiating the definitions of $\sin az$ and $\cos az$ shows that

$$\frac{d[\sin az]}{dz} = a \cos z \quad \text{and} \quad \frac{d[\cos az]}{dz} = -a \sin az \tag{36}$$

for all z, so $\sin az$ and $\cos az$ are entire functions.

By definition

$$\tan z = \frac{\sin z}{\cos z}, \tag{37}$$

so replacing z by az followed by an application of (19) together with results (35) and (36) gives

$$\frac{d[\tan az]}{dz} = \frac{a}{\cos^2 az} = a \sec^2 az, \tag{38}$$

provided $\cos az \neq 0$, so $\tan z$ is *not* an entire function.

The functions sec az, csc az, and cot az are defined in the usual manner as

$$\sec az = \frac{1}{\cos az}, \quad \csc az = \frac{1}{\sin az}, \quad \text{and} \quad \cot az = \frac{1}{\tan az}, \tag{39}$$

with the derivatives

$$\frac{d}{dz}[\sec az] = a \sec az \tan az, \quad \frac{d}{dz}[\csc az] = -a \csc az \cot az, \quad \text{and}$$
$$\frac{d}{dz}[\cot az] = -a \csc^2 az. \tag{40}$$

Summary of derivatives of elementary complex functions

1. $\dfrac{d}{dz}[z^n] = nz^{n-1}$, for $n = 0, \pm 1, \pm 2, \ldots,$ and $z \neq 0$ when $n < 0$.
2. $\dfrac{d}{dz}[e^{az}] = ae^{az}$, for all a and z.
3. $\dfrac{d}{dz}[\sinh az] = a \cosh az$, for all a and z.
4. $\dfrac{d}{dz}[\cosh az] = a \sinh az$, for all a and z.
5. $\dfrac{d}{dz}[\tanh az] = a\,\mathrm{sech}^2 az$, for $\cosh az \neq 0$.
6. $\dfrac{d}{dz}[\mathrm{sech}\, az] = -a\,\mathrm{sech}\, az \tanh az$, for $\cosh az \neq 0$.
7. $\dfrac{d}{dz}[\mathrm{csch}\, az] = -a\,\mathrm{csch}\, az \coth az$, for $\sinh az \neq 0$.
8. $\dfrac{d}{dz}[\coth az] = -a\,\mathrm{csch}^2 az$, for $\sinh az \neq 0$.
9. $\dfrac{d}{dz}[\sin az] = a \cos az$, for all a and z.
10. $\dfrac{d}{dz}[\cos az] = a \sin az$, for all a and z.
11. $\dfrac{d}{dz}[\tan az] = a \sec^2 az$, for $\cos az \neq 0$.
12. $\dfrac{d}{dz}[\sec az] = a \sec az \tan az$, for $\cos az \neq 0$.
13. $\dfrac{d}{dz}[\csc az] = -a \csc az \cot az$, for $\sin az \neq 0$.
14. $\dfrac{d}{dz}[\cot az] = -a \csc^2 az$, for $\sin az \neq 0$.

Summary

After the definitions of a limit and the continuity of a complex function $f(z)$, its derivative $f'(z)$ was defined. The Cauchy–Riemann conditions were shown to ensure the differentiability of a complex function, and a function that has a continuous derivative throughout some part of the complex plane was called an analytic function. Derivatives of the complex exponential, complex hyperbolic, and complex trigonometric functions were derived.

EXERCISES 13.2

In Exercises 1 through 4 find the real and imaginary parts of the functions and locate any points where they are discontinuous.

1. $f(z) = z^3 + 4z^2 - 3z + 1$.

2. $f(z) = 1 + z^2 + z\bar{z}$.

3. $f(z) = z/(1 + z^2)$.

4. $f(z) = (z - 1)/(z + 1)$.

In Exercises 5 through 8, use the definition of a derivative given in (16) to determine if the given function $f(z)$ is differentiable and, when it is, to find $f'(z)$. Locate any points where the derivative is not defined.

5. $f(z) = z^3 + z + 1$.

6. $f(z) = 3 + \bar{z}$.

7. $f(z) = 1/(1 + z)$.

8. $f(z) = 1/(a + z)^2$, with a a complex constant.

In Exercises 9 through 12 use the Cauchy–Riemann equations to show that the given function $f(z)$ is differentiable. Use the result to find $f'(z)$ both in its cartesian form and as a function of z, and locate any points where the derivative is not defined.

9. $f(z) = z^3$.

10. $f(z) = 1/(4 + z)$.

11. $f(z) = z + 1/z$.

12. $f(z) = 1/(z^2 + 1)$.

In Exercises 13 through 16 use the definitions of complex hyperbolic functions to establish the stated identities.

13. $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$, and deduce that $\sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$.

14. $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$, and deduce that $\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$.

15. $\cosh^2 z - \sinh^2 z = 1$ and $\tanh^2 z = 1 - \operatorname{sech}^2 z$.

16. $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$.

In Exercises 17 through 20 use the definitions of the complex trigonometric functions to establish the stated identities.

17. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$, and deduce that $\sin(x \pm iy) = \sin x \cosh y \pm i \cos x \sinh y$.

18. $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$, and deduce that $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$.

19. $\sin^2 z + \cos^2 z = 1$ and $\tan^2 z = \sec^2 z - 1$.

20. $\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$.

In Exercises 21 through 29 use the method of Example 13.10 to find the roots of the given equations.

21. $\sin z = 0$.

22. $\cos z = 0$.

23. $\sinh z = 0$.

24. $\sin z = \cosh 2$.

25. $\cos z = -\cosh 3$.

26. $\sin z = 7$.

27. $\sinh z = i \cosh 2$.

28. $\cos z = -i \sinh 5$.

29. $\tanh z = 0$.

In Exercises 30 and 31, locate the points where the given functions are not analytic in the specified domains.

30. (a) $\sec z$ for $|z| < 3$. (b) $\sin z/(1 + z^2)$ for $|z| < 2$. (c) $\cos z/(1 + z)^2$ for $|z| < \pi$.

31. (a) $\csc z/(z^2 - 3i)$ for $|z| < 4$. (b) $1/(z^4 + 16)$ for $|z| < 3$. (c) $|z| \tan z$ for $|z| < 2$.

32. Show that $f(z) = \cosh 2z$ satisfies the Cauchy–Riemann equations for all z. Hence, find $f'(z)$ both in its cartesian form and as a function of z.

33. Show that $f(z) = \sin 3z$ satisfies the Cauchy–Riemann equations for all z. Hence, find $f'(z)$ both in its cartesian form and as a function of z.

34. Show that $f(z) = 1/\sinh z$ satisfies the Cauchy–Riemann equations for all z other than at the zeros of $\sinh z$. Hence, find $f'(z)$ both in its cartesian form and as a function of z.

35. Use the change of variable from the cartesian coordinates (x, y) to the polar coordinates (r, θ) given by $x = r\cos\theta$ and $y = r\sin\theta$ to show that the **polar form** of the **Cauchy–Riemann equations** for a single-valued analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ is

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

36. Use the change of variable from the cartesian coordinates (x, y) to the polar coordinates (r, θ) given by $x = r\cos\theta$ and $y = r\sin\theta$ to show that the derivative of a single-valued analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ is given by

$$f'(z) = \left(\cos\theta\frac{\partial u}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + i\left(\cos\theta\frac{\partial v}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial v}{\partial \theta}\right).$$

Explain why, when $f(z)$ is a single valued analytic function, this last result can be expressed as a function of z by setting $\theta = 0$ and replacing r by z.

37. Set $z = re^{i\theta}$ in $f(z) = z + 1/z$ and use the polar form of the Cauchy–Riemann equations given in Exercise 35 to show that $f(z)$ is differentiable for $z \neq 0$. Use the result of Exercise 36 to find $f'(z)$ as a function of z.

38. Set $z = re^{i\theta}$ in $f(z) = z^2 - 1/z^2$ and use the polar form of the Cauchy–Riemann equations given in Exercise 35 to show $f(z)$ is differentiable for $z \neq 0$. Use the result of Exercise 36 to find $f'(z)$ as a function of z.

39. Use the polar form of the Cauchy–Riemann equations given in Exercise 35 to verify that

$$f(z) = (3r^3\cos 3\theta + r\cos\theta + 1) + i(3r^3\sin 3\theta + r\sin\theta)$$

is an entire function, and then use the result of Exercise 36 to express $f'(z)$ as a function of z. Confirm that $f(z)$ is an entire function by first expressing $f(z)$ as a function of z and then differentiating the result.

40. Repeat Exercise 39 using

$$f(z) = \left(r^2\cos 2\theta - \frac{2}{r^2}\cos 2\theta + r\cos\theta\right) + i\left(r^2\sin 2\theta + \frac{2}{r^2}\sin 2\theta + r\sin\theta\right).$$

13.3 Harmonic Functions and Laplace's Equation

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D, and let functions $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives with respect to x and y. Then it is known from elementary calculus (see Theorem 1.3) that the mixed partial derivatives of $u(x, y)$ and $v(x, y)$ must be equal, so $\partial^2u/\partial x\partial y = \partial^2u/\partial y\partial x$ and $\partial^2v/\partial x\partial y = \partial^2v/\partial y\partial x$.

Differentiating the first Cauchy–Riemann equation in Theorem 13.1 partially with respect to x gives

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y}\right] \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y\partial x},$$

and differentiating the second Cauchy–Riemann equation in Theorem 13.1 partially with respect to y gives

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] = -\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}\right] \quad \text{or} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\partial y}.$$

Adding these two results and using the equality of mixed derivatives show that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.} \tag{41}$$

Had the first equation been differentiated partially with respect to y and the second partially with respect to x, addition of the results would have given

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.} \tag{42}$$

Results (41) and (42) show that both the real and imaginary twice differentiable parts of an analytic function satisfy the same second order *partial differential equation*. The partial differential equation

$$\boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0} \tag{43}$$

the Laplace equation, harmonic functions, and the Laplacian

is called the **Laplace equation**, and any function Φ that satisfies Laplace's equation is called a **harmonic function**. Thus, both $u = \text{Re}\{f(z)\}$ and $v = \text{Im}\{f(z)\}$ are harmonic functions, and they are defined throughout the domain D. We now define the symbol

Δ, pronounced "Laplacian," as

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{44}$$

Then Δ is a **differential operator**, and as it stands (44) is *not* a function because it only describes a *differentiation operation*. However, when the operator Δ acts on a suitably differentiable function $\Phi(x, y)$, indicated by placing the function $\Phi(x, y)$ immediately after the symbol Δ, the result $\Delta\Phi$ becomes a *function*. As the Laplace equation in (43) can be written as $\Delta\Phi = 0$, the symbol Δ defined in (44) is called the **Laplacian operator in two dimensions**, and $\Delta\Phi$ is called the **Laplacian** of Φ. Consequently, a function Φ will be *harmonic* if its Laplacian is zero.

When $f(z)$ is an analytic function with $u(x, y) = \text{Re}\{f(z)\}$ and $v(x, y) = \text{Im}\{f(z)\}$, the function $v(x, y)$ is called the **harmonic conjugate** of $u(x, y)$ and, conversely, $u(x, y)$ is called the **harmonic conjugate** of $v(x, y)$. It is important to recognize that two functions $U(x, y)$ and $V(x, y)$ that are harmonic can only be *harmonic conjugates* if U and V satisfy the Cauchy–Riemann equations.

harmonic conjugates

EXAMPLE 13.12

Given $f(z) = \sin z$ and $g(z) = \cos z$, find the harmonic conjugate functions $u_1(x, y) = \text{Re}\{f(z)\}$ and $v_1(x, y) = \text{Im}\{f(z)\}$ associated with $f(z)$, and the harmonic conjugate functions $u_2(x, y) = \text{Re}\{g(z)\}$ and $v_2(x, y) = \text{Im}\{g(z)\}$ associated with $g(z)$. Verify that u_1, v_1, u_2, and v_2 are harmonic functions and show that the complex function $F(z) = u_1(x, y) + iv_2(x, y)$ is *not* analytic, and so $u_1(x, y)$ is not the harmonic conjugate of $v_2(x, y)$.

Solution As $f(z) = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, writing $f(z) = u_1 + iv_1$ we see that $u_1 = \sin x \cosh y$ and $v_1 = \cos x \sinh y$. The functions u_1 and v_1 are harmonic conjugate functions because straightforward differentiation confirms that u_1 and v_1 satisfy the Cauchy–Riemann equations. To verify that u_1 and v_1 are harmonic functions, it is necessary to show that each satisfies Laplace's equation. Differentiation gives

$$\frac{\partial^2 u_1}{\partial x^2} = -\sin x \cosh y \quad \text{and} \quad \frac{\partial^2 u_1}{\partial y^2} = \sin x \cosh y,$$

so

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad \text{or} \quad \Delta u_1 = 0,$$

confirming that u_1 is a harmonic function. The fact that v_1 is harmonic follows in similar fashion.

As $g(z) = \cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$, setting $g(z) = u_2 + iv_2$ shows that $u_2 = \cos x \cosh y$ and $v_2 = -\sin x \sinh y$. These are harmonic conjugate functions because they also satisfy the Cauchy–Riemann equations.

Although the functions $u_1(x, y) = \sin x \cosh y$ and $v_2(x, y) = -\sin x \sinh y$ forming the real and imaginary parts of $F(z) = u_1(x, y) + iv_2(x, y)$ are both harmonic, $\partial u_1/\partial x \neq \partial v_2/\partial y$, and $\partial u_1/\partial y \neq -\partial v_2/\partial x$, showing that $F(z)$ does not satisfy the Cauchy–Riemann equations, and so $F(z)$ is *not* analytic and $u_1(x, y)$ and $v_2(x, y)$ are *not* harmonic conjugates. ■

In (44) the Laplacian operator is expressed in its cartesian form, but if the cartesian coordinates (x, y) are changed to the polar coordinates (r, θ) by means of the transformation $x = r\cos\theta$ and $y = r\sin\theta$, the change of variable formulas from elementary calculus (see Theorem 1.11) shows that the Laplacian operator takes on the form

Laplacian in polar coordinates

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{45}$$

This means that when polar coordinates are used to express z in the form $z = re^{i\theta}$, and a single-valued analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ is considered, the functions $u(r, \theta)$ and $v(r, \theta)$ will each be harmonic, so

$$\begin{aligned} \Delta u(r, \theta) &\equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{and} \\ \Delta v(r, \theta) &\equiv \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0. \end{aligned} \tag{46}$$

It follows that $u(r, \theta)$ will be the harmonic conjugate of $v(r, \theta)$ and, conversely, $v(r, \theta)$ will be the harmonic conjugate of $u(r, \theta)$.

EXAMPLE 13.13

Set $z = re^{i\theta}$ in $f(z) = z + 1/z$, and by showing that when $z \neq 0$ the function $f(z)$ satisfies the polar form of the Cauchy–Riemann equations given in Exercise 37 of Exercise set 13.2, confirm that $f(z)$ is analytic when $z \neq 0$. Verify that the functions $u(r, \theta) = \mathrm{Re}\{f(z)\}$ and $v(r, \theta) = \mathrm{Im}\{f(z)\}$ are harmonic functions.

Solution $f(z) = z + 1/z = re^{i\theta} + \frac{1}{r}e^{-i\theta} = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$, and so

$$u(r, \theta) = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta.$$

Routine differentiation confirms that u and v satisfy the polar form of the Cauchy–Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad \text{for } r \neq 0,$$

so $f(z)$ is analytic for $z \neq 0$.

Straightforward differentiation shows that u and v satisfy the polar form of Laplace's equation and so are harmonic when $z \neq 0$. ■

In applications of complex analysis, as in Section 17.2 when solving a boundary value problem for the two-dimensional steady state temperature distribution in a solid, it can happen that a harmonic function $\Phi(x, y)$ is known, but it is required to find its harmonic conjugate $\Psi(x, y)$ so an analytic function $F(z) = \Phi(x, y) + i\Psi(x, y)$ can be constructed. The function $\Psi(x, y)$ can be found by making use of the Cauchy–Riemann equations that must be satisfied simultaneously by both $\Phi(x, y)$ and $\Psi(x, y)$.

We now show how an analytic function $f(z) = u(x, y) + iv(x, y)$ can be constructed when either one of the harmonic conjugate functions $u(x, y)$ or $v(x, y)$ is

how to find an analytic function from one of its harmonic conjugate functions

known. Let us suppose that a harmonic function $u(x, y)$ is known. Then from the first of the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \tag{47}$$

where the expression on the right can be found by differentiation of the known function $u(x, y)$.

If we reverse the process by which $\partial u/\partial x$ was found, by integrating (47) with respect to y while keeping x constant, we obtain

$$v(x, y) = \int \frac{\partial u}{\partial x} dy + g(x) + a, \tag{48}$$

where $g(x)$ is an arbitrary function of x and a is an arbitrary real integration constant.

The inclusion of the arbitrary function $g(x)$ in (48) in addition to the usual arbitrary integration constant a is necessary to make the expression on the right the most general antiderivative that can be obtained when (47) is integrated with respect to y while holding x constant. The result can be checked by differentiating (48) partially with respect to y to return to (47), because after differentiation the first term on the right reduces to $\partial u/\partial x$ and the remaining terms vanish because $\partial\{g(x) + a\}/\partial y \equiv 0$. It is obvious that (48) can be simplified by including the arbitrary constant a in the arbitrary function $g(x)$, but in applications it is usually better to retain it explicitly as in (48).

If we rewrite the second Cauchy–Riemann equation as

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

the term on the right is again known by differentiation of $u(x, y)$. Integration of this equation with respect to x while keeping y constant gives

$$v(x, y) = -\int \frac{\partial u}{\partial y} dx + h(y) + b, \tag{49}$$

where now $h(y)$ is an arbitrary function of y and b is an arbitrary real integration constant.

Expressions (48) and (49) must be identical, so $g(x)$ in (48) must be identified with any functions on the right of (49) that only involve x, and $h(y)$ in (49) must be identified with any functions on the right of (48) that only involve y, whereas the arbitrary constants must be equal, so $b = a$. The required analytic function is then seen to be

$$f(z) = u(x, y) + iv(x, y) + ia. \tag{50}$$

An analogous argument shows how if $v(x, y)$ is known instead of $u(x, y)$, then

$$u(x, y) = \int \frac{\partial v}{\partial y} dx + H(y) + C, \tag{51}$$

and

$$u(x, y) = -\int \frac{\partial v}{\partial x} dy + G(x) + D, \tag{52}$$

with $H(y)$ an arbitrary function of y, $\mathrm{G}(x)$ an arbitrary function of x, and C and D arbitrary real integration constants. The form of argument used to arrive at (50)

then shows that the required analytic function is

$$f(z) = u(x, y) + iv(x, y) + D. \tag{53}$$

It is to be expected that the analytic function $f(z)$ can only be determined up to an arbitrary additive constant, because a constant is always a solution of Laplace's equation. In applications, either the constant occurring in (50) or (53) is unimportant, and so can be set equal to zero, or, if needed, it must be determined by some additional condition satisfied by the analytic function $f(z)$.

To understand why the introduction of an arbitrary additive constant to a solution of Laplace's equation causes no difficulties in applications, it is only necessary to consider problems like the determination of a steady state temperature distribution or an electrostatic potential distribution. In these cases, and in others of a similar type, what matters is the temperature or potential *difference*, rather than their absolute values, so the arbitrary additive constant simply represents a convenient reference level from which all other temperatures or potentials are measured.

EXAMPLE 13.14 Given $u(x, y) = x^2 - y^2 + x - y$, find its harmonic conjugate $v(x, y)$ and construct the most general analytic function $f(z)$ such that $u(x, y) = \text{Re}\{f(z)\}$.

Solution First it is necessary to check that $u(x, y)$ is a harmonic function, and this can be seen from the fact that

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2, \quad \text{and so} \quad \Delta u = 0.$$

As $\partial u/\partial x = 2x + 1$, result (48) becomes

$$v(x, y) = \int (2x + 1)dy + g(x) + a,$$

so

$$v(x, y) = 2xy + y + g(x) + a.$$

Using the fact that $\partial u/\partial y = -2y - 1$, result (49) becomes

$$v(x, y) = -\int (-2y - 1)dx + h(y) + b,$$

so

$$v(x, y) = 2xy + x + h(y) + b.$$

These two expressions for $v(x, y)$ will be identical if $g(x) = x$, $h(y) = y$, and $a = b$, so

$$v(x, y) = 2xy + x + y + a,$$

with a an arbitrary real constant. The cartesian form of the required analytic function is

$$f(z) = x^2 - y^2 + x - y + i(2xy + x + y) + ia.$$

Setting $y = 0$ and replacing x by z to convert this to an analytic function in terms of z shows that

$$f(z) = z^2 + (1 + i)z + ia.$$ ■

For more information and examples involving limits, continuity, differentiability, and elementary functions of a complex variable, see any one of references [6.1] to [6.4] and [6.6] to [6.9].

Summary

Harmonic functions were introduced as solutions of Laplace's equation, and in an analytic function $f(z) = u + iv$ the functions u and v were shown to be harmonic. The functions u and v in an analytic function were called harmonic conjugates, and it was shown how to reconstruct $f(z)$ when either of its harmonic conjugates u or v is known.

EXERCISES 13.3

In Exercises 1 through 10, verify that the given function is harmonic, and find its harmonic conjugate. Use the result to construct the most general analytic function $f(z)$ as a function of z.

1. $u(x, y) = x^3 - 3xy^2 + 2x + y$.
2. $u(x, y) = e^{2x}(x\cos 2y - y\sin 2y)$.
3. $v(x, y) = e^{-y}(y\cos x + x\sin x) + 2x$.
4. $v(x, y) = x^3 - 3xy^2 + x + y$.
5. $v(x, y) = y\sinh 2x\cos 2y + x\cosh 2x\sin 2y$.
6. $u(x, y) = \sin 3x\cosh 3y - 2x^2 + 2y^2$.
7. $u(x, y) = x\cos 3x\cosh 3y + y\sin 3x\sinh 3y$.
8. $v(x, y) = e^{-y}(3\cos x + 2\sin x) - 5y$.
9. $u(r, \theta) = r\cos\theta + 2r^2\cos 2\theta + r^2\sin 2\theta$.
10. $v(r, \theta) = r\sin\theta + \dfrac{1}{r^2}\sin 2\theta$.
11. Show that $u(x, y) = xy$ and $v(x, y) = x^3 - 3xy^2$ are both harmonic functions, but they are not harmonic conjugates.
12. Show that $u(x, y) = -x^2 + y^2 + 2xy$ and $v(x, y) = x^3 - 3xy^2 + 3x^2y - y^3$ are both harmonic functions, but they are not harmonic conjugates.
13. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, and either $u(x, y)$ = constant or $v(x, y)$ = constant, then $f(z)$ = constant in D. Does this result remain true if $f(z)$ is not analytic? If not, explain why and give an example.
14. Given that $\Phi(x, y) = a(1 - 2x^2 + 2y^2)\sin 2x\cosh 2y + 4xy\cos 2x\sinh by$, find $\Delta\Phi$, and hence determine the values of the constants a and b that make Φ a harmonic function.
15. Given that $\Phi(x, y) = (2 + ax^2 - y^2)\sinh x\cos y + bxy\cosh x\sin y$, find $\Delta\Phi$, and hence determine the values of the constants a and b that make Φ a harmonic function.

13.4 Elementary Functions, Inverse Functions, and Branches

The elementary analytic functions considered so far have been polynomials, rational functions (quotients of polynomials), the exponential function, and the trigonometric and hyperbolic functions. All of these have involved the fundamental idea that for f to be a *function*, one point in the domain of definition of f must correspond to one point in the range of f. If the domain of definition of f is D and its range is Ω and we set $w = f(z)$, then z is any point of D and w is the corresponding point in Ω.

In addition to the connection between the domain D of f and its range Ω, expressed by the functional relationship $w = f(z)$, it is also necessary to be able to proceed in the reverse direction, by starting with a point w in Ω and finding the point or points z in D to which it corresponds. This is the **inverse** relationship involving f, and it is convenient to represent it by writing $z = f^{-1}(w)$. For this inverse relationship to be a *function* it is necessary that f^{-1} has the property that to every w in Ω there corresponds only one z in D.

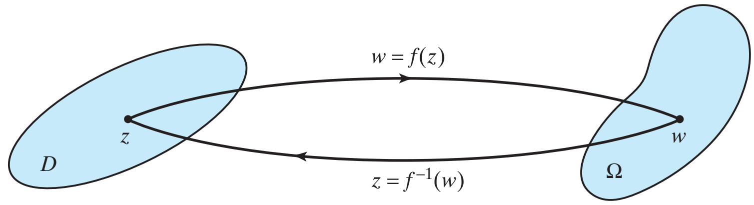


FIGURE 13.6 If f is a one-one analytic function, then $f(f^{-1}(w)) = w$ and $f^{-1}(f(z)) = z$.

In general, if the analytic function

$$w = f(z) \tag{54}$$

maps its domain of definition D onto a domain Ω and, in addition, if to each w in Ω there corresponds only one z in D given by $z = f^{-1}(w)$, the function f is **one-one**, and the function f^{-1} is called the **inverse** of the function f. This means that if an analytic function f is one-one, then

inverse function

$$f(f^{-1}(w)) = w \quad \text{and} \quad f^{-1}(f(z)) = z. \tag{55}$$

The relationship between a one-one analytic function f and its inverse f^{-1} is shown diagramatically in Fig. 13.6.

Let us now show that if f is a one-one analytic function defined for z in D, and

$$f'(z) \neq 0,$$

then the inverse function $z = f^{-1}(w)$ is analytic in Ω. This result is easily proved by using the definition of differentiability and setting $z + h = f^{-1}(w + k)$, so that $w + k = f(z + h)$. Differentiation $f^{-1}(w)$ gives

$$\frac{d}{dw}[f^{-1}(w)] = \lim_{k\to 0}\left[\frac{f^{-1}(w+k) - f^{-1}(w)}{k}\right] = \lim_{h\to 0}\left[\frac{h}{f(z+h) - f(z)}\right]$$
$$\times \lim_{h\to 0}\left\{1\Big/\left[\frac{f(z+h) - f(z)}{h}\right]\right\} = 1/f'(z).$$

Then, as by hypothesis $f'(z) \neq 0$, it follows that $d[f^{-1}(w)]/dw$ exists and is unique in Ω, so $f^{-1}(w)$ is analytic in Ω, and the result is proved.

One of the simplest examples of a one-one analytic function is provided by the **linear function** $w = az + b$ with $a \neq 0$, because this is analytic throughout the z-plane and maps every point of it one-one onto the w-plane, and the inverse function $z = (w - b)/a$ is also analytic throughout the w-plane.

A slightly more complicated example of a one-one analytic function is the **linear fractional function**

linear fractional function

$$w = \frac{az + b}{cz + d} \tag{56}$$

that is analytic in any domain D in the z-plane in which $z \neq -d/c$, because then dw/dz is defined throughout D. Solving the linear fractional function in (56) for z

shows the inverse function to be given by

$$z = \frac{b - wd}{wc - a}.$$

This inverse function is also analytic, and it maps any domain Ω in the w-plane where $w \neq a/c$ onto a corresponding domain D in the z-plane. The condition $w \neq a/c$ ensures the analyticity of the inverse function because then dz/dw is defined and unique throughout Ω.

Inverse functions associated with functions as simple as $w = z^2$, $w = \exp z$, and the hyperbolic and trigonometric functions require special attention because these functions exhibit periodicity in the complex plane. This periodicity has the effect that although one z corresponds to one w, the converse is not true because one w corresponds to more than one z, and often to infinitely many values of z. To overcome this difficulty it is necessary to confine z to a restricted domain in the z-plane to make the relationship between the restricted domain in the z-plane and the w-plane one-one. To illustrate this approach we will consider the function $w = z^n$, and its inverse the ***n*th root function** $z = w^{1/n}$, where n is a positive integer.

***n*th root function**

When expressed in polar form by writing $w = \rho e^{i\phi}$ and $z = re^{i\theta}$, with $\theta = \text{Arg}\, z$, the function $w = z^n$ becomes $w = r^n e^{in\theta}$. So, as the argument of z is multiplied by n, any domain in the z-plane in the form of a sector with angle $2\pi/n$ centered on the origin will be mapped onto the entire w-plane, with the result that the function $w = z^n$ will map the entire z-plane onto the w-plane n times. Consequently, although one z corresponds to one w, the inverse operation $z = w^{1/n}$ will map n different values of w onto one point in the z-plane.

As it stands, the inverse formula $z = w^{1/n}$ represents many functions and so does *not* define a single function. To overcome this problem we divide the z-plane into n equal sectors $D_0, D_1, \ldots, D_{n-1}$, each centered on the origin, with D_k defined as the sector given by

$$(2k-1)\frac{\pi}{n} < \theta < (2k+1)\frac{\pi}{n}, \quad r > 0, \quad \text{for } k = 0, 1, 2, \ldots, n-1. \tag{57}$$

If we restrict $z = re^{i\theta}$ to any one of the sectors D_k, the function $w = z^n$ will map the sector D_k once onto the entire w-plane with the exception of points on the negative real axis up to and including the point at the origin. Conversely, when z is restricted to D_k, any point in the w-plane not on the negative real axis or at the origin will be mapped once by the function $z = w^{1/n}$ onto the sector D_k. The deletion of the points on the negative real axis up to and including the origin is called a **cut** in the w-plane.

Let ψ be such that $-\pi/n < \psi < \pi/n$, then in the kth sector D_k, $\theta = 2k\pi/n + \psi$ for $k = 0, 1, \ldots, n-1$. Using the polar representations for w and z by setting $z = re^{i\psi}$ and $w = \rho e^{i\phi}$ allows $w = z^n$ to be written

$$\rho e^{i\phi} = r^n \exp\left[in\left(\frac{2k\pi}{n} + \psi\right)\right],$$

so equating moduli and arguments we have

$$\rho = r^n \quad \text{and} \quad \phi = 2k\pi + n\psi,$$

showing that

$$r = \rho^{1/n} \quad \text{and} \quad \phi/n = 2k\pi/n + \psi,$$

where $\rho^{1/n}$ is the numerical value of the nth root of the positive real number ρ.

Solving for z in terms of w shows that the cut w-plane is mapped one-one onto the sector D_k by

$$z = \rho^{1/n}\left\{\cos\left[\frac{2k\pi}{n} + \psi\right] + i\sin\left[\frac{2k\pi}{n} + \psi\right]\right\}, \quad k = 0, 1, \ldots, n-1. \tag{58}$$

branch, principal branch, and branch cut

Each of the n different solutions in (58) is called a **branch** of the nth root function, and the branch corresponding to $n = 0$ is called the **principal branch**. The *cut* in the w-plane separating one branch from another is called a **branch cut**. So the principal branch of the nth root function $z = w^{1/n}$ is

$$z = \rho^{1/n}[\cos\psi + i\sin\psi], \quad \text{with } -\pi/n < \psi \leq \pi/n. \tag{59}$$

The mapping of the sector D_0 onto the cut w-plane by $w = z^3$ and of the cut w-plane onto the z-plane by the principal branch of the cube root function $z = w^{1/3}$ is shown in Fig. 13.7, where shading has been used to show how different areas correspond. The mapping of D_1 onto the cut w-plane by $w = z^3$ and of the cut w-plane onto the z-plane by the second branch $(k = 1)$ of the cube root function is shown in Fig. 13.8, where shading has again been used to show how different areas correspond.

When it is necessary to consider the nth root function as a function of z, and not merely as the inverse of the power function $w = z^n$, all that is necessary in (59) is to interchange z and w and their associated moduli and arguments, leading to the corresponding result for the function $w = z^{1/n}$.

The complex exponential function $w = e^z$ has been defined as

$$e^z = e^x(\cos y + i\sin y),$$

so as $\sin y$ and $\cos y$ are periodic with period 2π, it can be seen that e^z is periodic with period $2\pi i$. This means that any strip of width 2π in the z-plane that is parallel to the real axis will be mapped onto the entire w-plane, with the exception of the

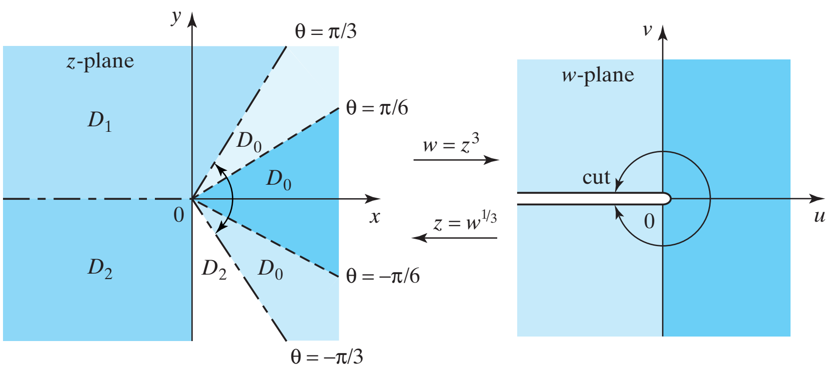


FIGURE 13.7 Mapping of sector D_0 in the z-plane onto the cut w-plane by $w = z^3$, and of the cut w-plane onto D_0 by the principal branch of $z = w^{1/3}$.

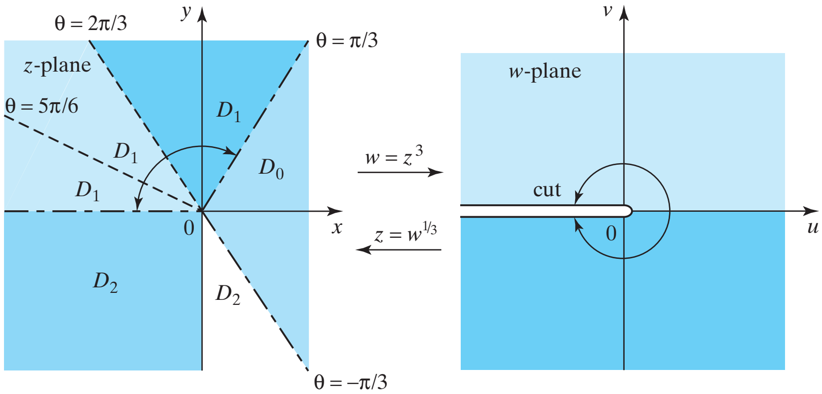


FIGURE 13.8 Mapping of sector D_1 in the z-plane onto the cut w-plane by $w = z^3$, and of the cut w-plane onto D_1 by the second branch of $z = w^{1/3}$.

origin. The origin must be excluded because $e^z \neq 0$ for any finite z, as may be seen from the fact that $|e^z| = e^x$, and e^x is never zero. The strip $-\pi < y \leq \pi$ is called the **fundamental strip** for the complex exponential function, and it is usual to refer to the complex plane from which the point at the origin has been removed as the **deleted complex plane**.

fundamental strip

Important properties of the complex exponential function are as follows:

(i) $e^{2\pi ni} = 1$ for n an integer, so $e^{z+2\pi ni} = e^z$ when n is an integer

(ii) If $w = e^z = \rho e^{i\phi}$, then $\rho = e^x$ and $\phi = \arg e^z = y \pm 2n\pi$ for all integers n

(iii) As $x = \ln \rho$, it follows that

$$z = x + iy = \ln \rho + i(\phi + 2n\pi)$$

and so

$$w = \exp[\ln |w| + i(\operatorname{Arg} w + 2n\pi)].$$

The inverse of the complex exponential function is the logarithmic function $\log z$, but the fact that any strip of width 2π parallel to the real axis in the z-plane will be mapped by $w = e^z$ onto the deleted w-plane means that the logarithmic function is infinitely many valued or, more simply, a **multivalued** function.

To make the multivalued complex logarithmic function into a one-one function, it is necessary to replace $\log z$ by a function with infinitely many branches, each corresponding to a strip of width 2π in the z-plane parallel to the real axis. The relationship between the planes then becomes one-one, because the exponential function will map a particular strip once onto the deleted w-plane and, conversely, a branch of the logarithmic function will map the deleted w-plane once onto the strip.

Using the symbol $\log z$ to denote the multifunction complex logarithmic function, and $\ln |z|$ to denote the natural logarithm of the real number $|z|$, we define the complex logarithm of the complex number z in the obvious manner as

$$\log z = \ln |z| + i \arg z, \quad \text{for } z \neq 0,$$

but $\arg z = \operatorname{Arg} z \pm 2n\pi$, with n an integer, so

$$\log z = \ln |z| + i(\operatorname{Arg} z \pm 2n\pi). \tag{60}$$

principal branch of the logarithmic function and principal value

Each of the expressions in (60) is to be regarded as a **branch** of the complex logarithmic function, and the branch for which $n = 0$ is taken to be the **principal branch** of the function. To avoid confusion, the principal branch is denoted by $\operatorname{Log} z$, where

$$\operatorname{Log} z = \ln|z| + i(\operatorname{Arg} z), \quad \text{with } z \neq 0 \quad \text{and} \quad -\pi < \operatorname{Arg} z \leq \pi. \tag{61}$$

For any given complex number z, the corresponding complex number defined by (61) is called the **principal value** of the logarithm of z.

EXAMPLE 13.15 Find $\log(1 + i\sqrt{3})$ and $\operatorname{Log}(1 + i\sqrt{3})$.

Solution Setting $z = 1 + i\sqrt{3}$ we find that that $|z| = 2$ and Arg $z = \pi/3$, and so $\log(1 + i\sqrt{3}) = \ln 2 + i(\frac{\pi}{3} + 2n\pi)$, and $\operatorname{Log}(1 + i\sqrt{3}) = \ln 2 + i\pi/3$. ■

Applying the polar form of the Cauchy–Riemann equations to $\operatorname{Log} z$ shows that it is an analytic function for $z \neq 0$, and the multivalued form of the complex logarithmic function possesses all the properties of the natural logarithmic function so, for example,

$$\log(z_1 z_2) = \log z_1 + \log z_2 \quad \text{and} \quad \log(z_1/z_2) = \log z_1 - \log z_2. \tag{62}$$

However, the restriction placed on the arguments of principal values means that these results do not always remain true when the multivalued logarithm $\log z$ is replaced by $\operatorname{Log} z$.

We are now in a position to generalize the power function $w = z^a$, where a is an arbitrary real number. To do this we write $w = z^a$ in the form

$$w = z^a = e^{a \operatorname{Log} z} = e^{a[\ln|z| + i(\operatorname{Arg} z + 2n\pi)]} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots,$$

and setting $z = re^{i\theta}$ this becomes

$$w = r^a\{\cos[a(\theta + 2n\pi)] + i\sin[a(\theta + 2n\pi)]\}. \tag{63}$$

inverse trigonometric and hyperbolic functions

We must now consider the behavior of the complex hyperbolic and trigonometric functions that map the complex z-plane more than once onto the w-plane, causing their inverses to be multivalued. To see how suitable branches can be introduced, we consider the typical example $w = \arcsin z$, which is the inverse of the function $z = \sin w$, so $\sin(\arcsin z) = z$. From the definition of the sine function,

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}},$$

so

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

Solving this quadratic equation for e^{iw} we find

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where the $\pm$ sign usually inserted in front of the square root has been omitted because the function $w = z^{1/2}$ implies that the square root function is two-valued.

Taking the complex logarithm of this result, we have

$$iw = \log[iz + (1 - z^2)^{1/2}],$$

and so

$$w = \arcsin z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{64}$$

Because of its branches the log function must be interpreted as many one-one functions, all with the same domain, but each branch having a different range.

Similar arguments applied to the other complex, trigonometric functions and to the complex hyperbolic functions show that

$$\arccos z = -i \log[z + i(1 - z^2)^{1/2}] \tag{65}$$

$$\arctan z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right) \tag{66}$$

$$\operatorname{arcsinh} z = \log[z + (1 + z^2)^{1/2}] \tag{67}$$

$$\operatorname{arccosh} z = \log[z + (z^2 - 1)^{1/2}] \tag{68}$$

$$\operatorname{arctanh} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right). \tag{69}$$

In each of the preceding cases, the branch of the inverse function involved is determined by the choice of branch in the square root and complex logarithmic function that appears on the right.

derivatives of inverse trigonometric and hyperbolic functions

Differentiation shows that:

$$\frac{d}{dz}[\arcsin z] = \frac{1}{(1 - z^2)^{1/2}} \tag{70}$$

$$\frac{d}{dz}[\arccos z] = \frac{-1}{(1 - z^2)^{1/2}} \tag{71}$$

$$\frac{d}{dz}[\arctan z] = \frac{1}{1 + z^2} \tag{72}$$

$$\frac{d}{dz}[\operatorname{arcsinh} z] = \frac{1}{(z^2 + 1)^{1/2}} \tag{73}$$

$$\frac{d}{dz}[\operatorname{arccosh} z] = \frac{1}{(z^2 - 1)^{1/2}} \tag{74}$$

$$\frac{d}{dz}[\operatorname{arctanh} z] = \frac{1}{1 - z^2}. \tag{75}$$

EXAMPLE 13.16

Show that the result obtained from (64) with $z = 1$ is consistent with the real variable trigonometric result $\arcsin 1 = (4n + 1)\pi/2$, for $n = 0, \pm 1, \pm 2, \ldots$.

Solution From (64), $\arcsin 1 = -i \log i$, but $i = \exp[i(\frac{\pi}{2} + 2n\pi)] = \exp[i(4n + 1)\frac{\pi}{2}]$, for $n = 0, \pm 1, \pm 2, \ldots$, and so

$$\arcsin 1 = -i \log i = -i\left[i(4n + 1)\frac{\pi}{2}\right] = (4n + 1)\frac{\pi}{2}, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

The principal value of this result, obtained by using the principal value $\operatorname{Log} z$ of $\log z$ corresponding to $n = 0$, is $\arcsin 1 = \pi/2$. ■

EXAMPLE 13.17 Find all the values of $\arcsin i$ and identify the one corresponding to the principal values of the square root and logarithmic functions.

Solution From (64), $\arcsin i = -i\log[-1+\sqrt{2}]$, but $2 = 2e^{2m\pi i}$, for $m = 0, \pm 1, \pm 2, \ldots$, so

$$\sqrt{2} = 2^{1/2}e^{m\pi i}, \quad \text{for } m = 0, \pm 1, \pm 2, \ldots.$$

As $e^{m\pi i}$ is either 1 or -1, according as m is even or odd, the value corresponding to the principal branch $(m = 0)$ is $\sqrt{2} = 2^{1/2}$, while the one corresponding to the second branch $(m = 1)$ is $\sqrt{2} = -2^{1/2}$, where $2^{1/2}$ denotes the *positive* square root of 2.

Case m = 0 (The principal branch): If the principal value of $\sqrt{2}$ is used, $-1+\sqrt{2} = 2^{1/2}-1$ is positive and $\arcsin i = -i\log(2^{1/2}-1)$, so writing $2^{1/2}-1 = (2^{1/2}-1)e^{2n\pi i}$, for $n = 0, \pm 1, \pm 2, \ldots$, shows that in this case

$$\arcsin i = -i\log(2^{1/2}-1) = 2n\pi - i\ln(2^{1/2}-1), \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

The value obtained for $\arcsin i$ depends on the choice of n, which in turn identifies the branch of the logarithmic function that is used to determine the value of $\log(2^{1/2}-1)$.

Case m = 1 (The second branch): If the second value of $\sqrt{2}$ is used, $-1-\sqrt{2} = -(2^{1/2}+1)$ is negative, so now we have $\arcsin i = -i\log[-(2^{1/2}+1)]$, but $-(2^{1/2}+1) = (2^{1/2}+1)e^{\pi i} = (2^{1/2}+1)e^{\pi i}e^{2n\pi i} = (2^{1/2}+1)e^{(2n+1)\pi i}$, for $n = 0, \pm 1, \pm 2, \ldots$. So $\log[-(2^{1/2}+1)] = \ln(2^{1/2}+1) + (2n+1)\pi i$, leading to the result

$$\arcsin i = (2n+1)\pi - i\ln(2^{1/2}+1).$$

The value of $\arcsin i$ obtained by using the principal values of the square root function $(m = 0)$ and the logarithmic function $(n = 0)$ is

$$\arcsin i = -i\ln(2^{1/2}-1).$$

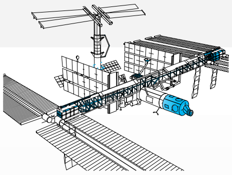
More information about inverse functions and branches can be found in references [6.1] to [6.4] and [6.6] to [6.9]. In particular, reference [6.4] provides valuable insight into the nature of the inverse of elementary functions of a complex variable.

EXERCISES 13.4

In Exercises 1 through 6 find all of the values of the given inverse functions and state the value obtained by using the principal value of the function or functions involved.

1. $\arccos 2i$.
2. $\operatorname{arccosh} 4i$.
3. $\arctan i$.
4. $\arctan 3i$
5. $\arctan\left(-\dfrac{2}{5}+\dfrac{1}{5}i\right)$.
6. $\operatorname{arctanh}\left(\dfrac{3}{7}+i\dfrac{2\sqrt{3}}{7}\right)$.
7. Show that $\arcsin z + \arccos z = \pi/2 + 2n\pi$.
8. Show that $u(x, y) = \ln(x^2+y^2)$ and $v(x, y) = \arctan(y/x)$ are analytic throughout the (x, y)-plane with the exception of the points on the imaginary axis.
9. Use the definition of $\operatorname{Log} z$ to show that it is discontinuous at $z = 0$, and also that it experiences a jump of πi across the negative real axis.
10. Use implicit differentiation on the function $z = \exp w$ to show that its inverse $w = \log z$ has the derivative

$$\frac{d}{dz}[\log z] = \frac{1}{z}, \quad \text{for } z \neq 0.$$



CHAPTER 13
TECHNOLOGY PROJECTS

Project 1

Finding how $w = az + b$ Maps a Given Curve in the z-Plane onto the w-Plane

This project explores how the two complex constants a and b in $w = az + b$ influence the way in which a curve in the z-plane is mapped by this function onto an image curve in the w-plane. This project anticipates some of the ideas that will be examined later in more detail in the chapter on conformal mapping.

Let $z(t) = x(t) + iy(t)$, with $x(t) = t(\pi - t)$, $y(t) = \sin(2t)$; and $0 \le t \le \pi$. Then as t increases from 0 to π, so the point $(x(t), y(t))$ in the z-plane with t as a parameter will describe a curve C_z in the z-plane. If $w(t) = az(t) + b$, with a and b complex numbers, each point of the curve C_z will be mapped by this function onto an image curve C_w in the w-plane. If we set $w(t) = u(t) + iv(t) = a(x(t) + iy(t)) + b$, the image C_w in the w-plane of the curve C_z in the z-plane is obtained by plotting the parametrically defined curve $(u(t), v(t))$.

Using the same length scales on the x- and y-axes, and also on the u- and v-axes, make computer plots of C_z and the corresponding image curves C_w given that: (i) $a = 2$, $b = 0$, (ii) $a = \frac{1}{2}$, $b = 1 + i$, (iii) $a = 2e^{i\pi/4}$, $b = 0$, (iv) $a = \frac{1}{3}e^{3\pi i/4}$, $b = -1 + i$.

Repeat the preceding numerical experiments using several values of a and b of your own choosing. Comment on the effect of $|a|$, Arg a, and b on the way the curve C_z is mapped onto the curve C_w.

Project 2

Another Example of Mapping by $w = az + b$

Repeat Project 1, but this time using $x(t) = t^3 - 2t$, $y(t) = 4 - t^2$, and $-2 \le t \le 2$.

Project 3

Finding an Analytic Function from One of Its Harmonic Conjugates

This project uses computer algebra to find an analytic function $f(z)$ when only its imaginary part is known in cartesian form.

Show that the function

$$v(x, y) = 3e^{2x}(x \sin 2y + y \cos 2y) + 2 \sin x \cosh y + 6x^2 y - 2y^3 + 4x + 3$$

is harmonic. Find its harmonic conjugate $u(x, y)$ and hence find the corresponding analytic function $f(z) = u + iv$ as a function of z, given that $f(0) = 3i$.

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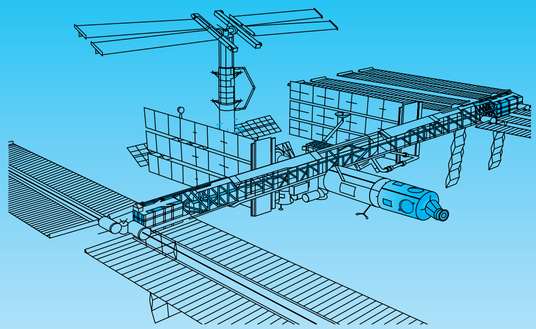
CHAPTER 14



Complex Integration

Both derivatives and integrals of analytic functions occur extensively in applications, so this chapter extends the results of Chapter 13 to include integration. As the integral of a complex function is evaluated either along or around a curve, the chapter starts by developing the concept of integration along a parametrically defined path or curve. It is then shown why, for the result to be independent of the path, the complex function must be an analytic function, that is, it must satisfy the Cauchy–Riemann equations.

Integrals of this type are called line integrals of complex functions, and when the path of integration is a closed curve in the form of a single loop, called either a simple curve or a Jordan curve, the integral is called a contour integral. The properties of line integrals are used to define indefinite integrals of complex functions, and fundamental results concerning contour integrals are proved and illustrated by example. Various properties of analytic functions are proved in the last section, including the important fundamental theorem of algebra that asserts that every polynomial of degree *n* has precisely *n* zeros, though some may be repeated.

14.1 Complex Integrals

path or contour

Complex integration involves integrating a single valued analytic function $f(z)$ in a given direction along a curve Γ in the complex z-plane. A non-self-intersecting curve Γ whose end points are not coincident is called a **path**, and paths are usually formed by joining straight line segments and arcs end to end. A closed path Γ in the form of a simple non-self-intersecting loop is called a **contour**. Paths and contours are usually specified parametrically by defining a general point z on Γ in the form

$$z = z(t) = x(t) + iy(t) \quad \text{for } t_0 \le t \le t_1, \tag{1}$$

where $x(t)$ and $y(t)$ are prescribed functions of the parameter t. Parametric representations are not unique, and in applications the simplest one is always used.

As t increases, so (1) determines the direction in which point z moves along Γ, and this direction is called the **sense** along the path or around the contour described

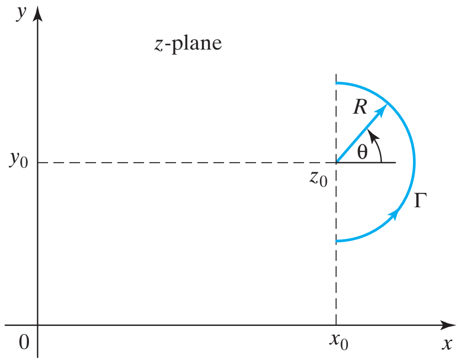


FIGURE 14.1 The semicircle Γ.

integration in positive sense

by the parametrization. In integration around a contour, the standard convention is that integration in the **positive sense** is taken to be in the **counterclockwise direction**.

An essential feature of the parametric description of a path or contour is that, in addition to its convenience when used in complex integration, it allows the description of curves that in a cartesian representation are many-valued. This is illustrated in the following example.

EXAMPLE 14.1

parametrizing a circular arc

Parametrize the semicircle Γ of radius R shown in Fig. 14.1 with its center at the point $z_0 = x_0 + iy_0$ in the z-plane.

Solution The cartesian representation of the semicircle Γ is $(x - x_0)^2 + (y - y_0)^2 = R^2$, with $x_0 \le x \le x_0 + R$, but this is ambiguous because when it is solved for y to give $y = y_0 + [R^2 - (x - x_0)^2]^{1/2}$, the square root operation makes y double valued. One way to overcome this difficulty is to use polar coordinates to describe a point (x, y) on a semicircle of radius R located at the origin by writing

$$x = R\cos\theta \quad \text{and} \quad y = R\sin\theta \quad \text{for} \quad -\pi/2 \le \theta \le \pi/2.$$

Each point on Γ is now described unambiguously in terms of the parameter θ. A shift of origin to the point (x_0, y_0) shows that the required parametric representation of Γ is

$$x = x_0 + R\cos\theta \quad \text{and} \quad y = y_0 + R\sin\theta, \quad -\pi/2 \le \theta \le \pi/2,$$

so

$$z(\theta) = x_0 + R\cos\theta + i(y_0 + R\sin\theta), \quad -\pi/2 \le \theta \le \pi/2.$$

In this representation, as θ *increases*, so z moves *counterclockwise* (positively) around the semicircle Γ. The choice of symbol for the parameter is immaterial, so the result could equally well be written

$$z(t) = x_0 + R\cos t + i(y_0 + R\sin t), \quad -\pi/2 \le t \le \pi/2.$$

Clearly this is not the only possible parametric description of Γ in terms of sines and cosines, because the change of variable $t = 1 + s$ gives the equivalent parametric description in terms of s

$$z(s) = x_0 + R\cos(1 + s) + i[y_0 + R\sin(1 + s)], \quad -\left(\frac{1}{2}\pi + 1\right) \le s \le \left(\frac{1}{2}\pi - 1\right).$$

Other parametric representations of this type can be found by making different changes of variable, provided only that the new argument of the sine and cosine functions increases monotonically from $-\pi/2$ to $\pi/2$.

Differentiation of $z(t)$ shows that the differential dz along Γ as t increases is

$$dz = (-R\sin t + iR\cos t)dt,$$

so if $dz = dx + i\,dy$, then

$$dx = -R\sin t\,dt \quad \text{and} \quad dy = R\cos t\,dt. \qquad \blacksquare$$

EXAMPLE 14.2

Let A and B be the points (3, 1) and (5, 7) in the z-plane. Parametrize the straight line segment AB in terms of parameter t so that (a) the sense is from A to B as t increases, and (b) the sense is from B to A as t increases.

Solution

(a) The cartesian equation of a straight line with gradient m passing through the point (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = m.$$

The gradient of the line segment AB is $m = (y_B - y_A)/(x_B - x_A) = (7-1)/(5-3) = 3$, so taking (3, 1) for the point (x_1, y_1) and substituting into the foregoing result shows that the straight line through AB in Fig. 14.2 has the equation

$$y = 3x - 8.$$

The line segment AB is obtained from the equation $y = 3x - 8$ by restricting x to $3 \le x \le 5$. To parametrize the line segment AB in terms of t, we set

$$x = t \quad \text{and} \quad y = 3t - 8, \quad \text{with} \quad 3 \le t \le 5,$$

so that

$$z(t) = t + i(3t - 8), \quad 3 \le t \le 5.$$

It is easily seen from this parametrization that an increase in t induces a *sense* along the line segment from A to B. Differentiation shows that the differential along

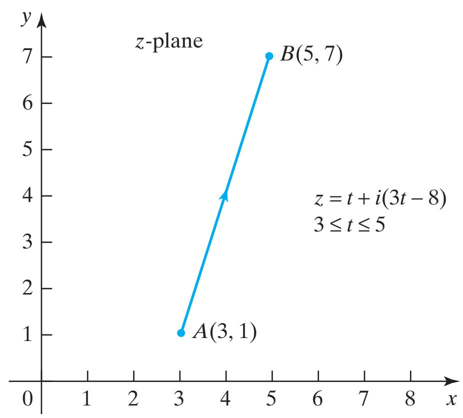


FIGURE 14.2 The line segment AB.

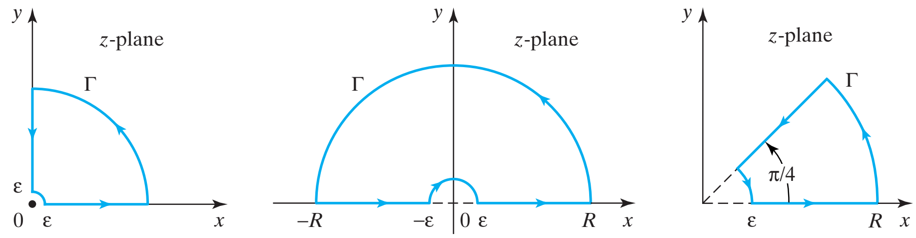


FIGURE 14.3 Some typical contours that arise in complex integration.

the line segment as t increases is

$$dz = dt + 3i\,dt \quad \text{so that } dx = dt \quad \text{and} \quad dy = 3dt.$$

(b) To reverse the sense along the line segment as t increases necessitates using a parameter that *decreases* as t *increases*. As the limits on t are $3 \le t \le 5$, this is most easily accomplished by setting $t = 5 - T$, because then $T = 0$ corresponds to $t = 5$ and $T = 2$ corresponds to $t = 3$. Substituting for t in the previous expression for $z(t)$ gives

$$z(T) = 5 - T + i(7 - 3T) \quad \text{for} \quad 0 \le T \le 2.$$

The differential dz along the line segment as T increases is now

$$dz = -dT - 3i\,dt, \quad \text{and so} \quad dx = -dT \quad \text{and} \quad dy = -3dT. \qquad \blacksquare$$

Typical examples of contours that arise in complex integration are shown in Fig. 14.3, in each of which the positive (counterclockwise) sense around the contour is shown by arrows.

line integral

The complex integral of an analytic function $f(z) = u(x, y) + iv(x, y)$ along the path Γ from A to B shown in Fig. 14.4, called a **line integral**, is denoted by $\int_{\Gamma_{AB}} f(z)dz$, where $dz = dx + i\,dy$. This integral is defined as

$$\begin{aligned}\int_{\Gamma_{AB}} f(z)dz &= \int_{\Gamma_{AB}} (u + iv)(dx + i\,dy) \\ &= \int_{\Gamma_{AB}} (u\,dx - v\,dy) + i\int_{\Gamma_{AB}} (v\,dx + u\,dy).\end{aligned} \tag{2}$$

contour integral

When Γ is a contour, and so is a simple non-self-intersecting loop, the integral $\int_\Gamma f(z)dz$ is called a **contour integral**, and this is sometimes indicated by writing $\oint_\Gamma f(z)dz$, though this notation will not be used here.

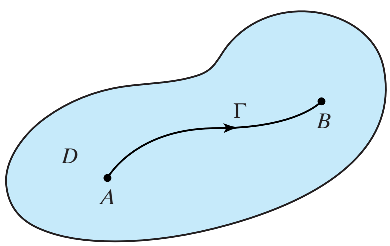


FIGURE 14.4 The path Γ for the line integral $\int_\Gamma f(z)dz$.

If the path Γ is parametrized as in (1), with A the point $z(t_0)$ and B the point $z(t_1)$, result (2) becomes

$$\int_{\Gamma_{AB}} f(z)dz = \int_{t_0}^{t_1} [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$
$$+ i\int_{t_0}^{t_1} [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]dt, \qquad (3)$$

where $x'(t) = dx/dt$ and $y'(t) = dy/dt$, showing that the evaluation of $\int_{\Gamma_{AB}} f(z)dz$ reduces to the calculation of two real integrals. It is usual to write (3) in the more concise form

$$\int_{\Gamma_{AB}} f(z)dz = \int_{t_0}^{t_1} f[z(t)]z'(t)dt. \qquad (4)$$

If in (4) the path Γ is constructed by joining end to end the successive paths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, the linearity of the ordinary definite integral allows $\int_{\Gamma_{AB}} f(z)dz$ to be written

$$\int_{\Gamma_{AB}} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \cdots + \int_{\Gamma_n} f(z)dz. \qquad (5)$$

The significance of the sense along a path is apparent from (4), because reversing the sense along Γ interchanges the limits on the integral and so changes the sign of the integral. Consequently, if Γ_-denotes the path Γ with its sense reversed, then

$$\boxed{\int_{\Gamma_-} f(z)dz = -\int_{\Gamma} f(z)dz.} \qquad (6)$$

As a complex integral involves the sum of two real integrals, the complex integral of a linear combination $Af(z) + Bg(z)$ of two analytic functions $f(z)$ and $g(z)$ shares the same linearity property as real integrals, and so

$$\boxed{\int_{\Gamma} \{Af(z) + Bg(z)\}dz = A\int_{\Gamma} f(z)dz + B\int_{\Gamma} g(z)dz,} \qquad (7)$$

where A and B are arbitrary complex constants.

The following Theorems contain important results that are used when working with complex integrals.

THEOREM 14.1

A fundamental inequality for complex integrals Let Γ be any path of finite length L, and let $f(z)$ be a complex function. Then the following inequality holds

(i) $$\boxed{\left|\int_{\Gamma} f(z)dz\right| \le \int_{\Gamma} |f(z)||dz|}$$

and

(ii) $$\boxed{\int_{\Gamma} |dz| = L.}$$

Proof

(i) It was shown in (3) that the real and imaginary parts of a complex line integral are both real integrals, so the complex line integral $\int_\Gamma f(z)dz$ can be defined in essentially the same way as a real definite integral. Let a sequence of points $z_0, z_1, \ldots, z_n$ lie along Γ, with z_0 at one end and z_n at the other. Then if $\Delta_k = z_k - z_{k-1}$, and ζ_k is any point on the straight line segment joining z_{k-1} and z_k, generalizing the definition of a real definite integral we have

$$\int_\Gamma f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\zeta_k)\Delta z_k,$$

when $|\Delta z_k| = |z_k - z_{k-1}| \to 0$ for all k as $n \to \infty$.

Taking the modulus of $\sum_{k=1}^{n} f(\zeta_k)\Delta z_k$ and making repeated use of the triangle inequality gives

$$\left|\sum_{k=1}^{n} f(\zeta_k)\Delta_k\right| \le \sum_{k=1}^{n} |f(\zeta_k)||\Delta_k|,$$

so proceeding to the limit as $n \to \infty$ this becomes

$$\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz|.$$

(ii) Setting $f(z) = 1$ in the result (i), and using the fact that $|dz| = [(dx)^2 + (dy)^2]^{1/2} = ds$, where ds is the element of arc length along Γ, we see that

$$\int_\Gamma |dz| = \int_\Gamma ds = L,$$

and the theorem is proved. ■

THEOREM 14.2

a useful estimate for the modulus of an integral

Estimating the modulus of an integral On a path Γ of finite length L, let $|f(z)|$ be bounded above by the positive real constant M, so that $|f(z)| \le M$ when z lies on Γ. Then

$$\left|\int_\Gamma f(z)dz\right| \le ML.$$

Proof The result follows directly from Theorem 14.1. Using the bound $|f(z)| \le M$ reduces (i) to

$$\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz| \le M\int_\Gamma |dz|,$$

and using (ii) this becomes

$$\left|\int_\Gamma f(z)dz\right| \le ML,$$

so the theorem is proved. ■

Because an upper bound of $|f(z)|$ is denoted by M, and the length of path Γ is denoted by L, this theorem is sometimes called the ML theorem.

EXAMPLE 14.3

Let the points A, B, and C at $(2, 2)$, $(6, 2)$, and $(6, 3)$, respectively, form a triangle as shown in Fig. 14.5. Take Γ_1 to be the path $AB + BC$, Γ_2 to be the path AC, and Γ_3 to

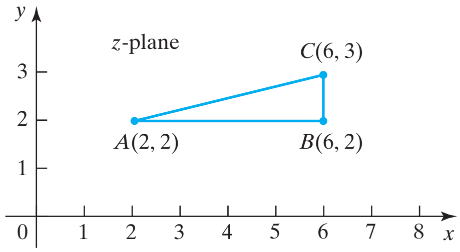


FIGURE 14.5 The points A, B, and C.

be the path $AB + BC + CA$, with the senses along the line segments indicated by the order of the letters. Set $f(z) = z$ and find the integrals $\int_{\Gamma_i} f(z)dz$, for $i = 1, 2, 3$. Verify Theorem 14.2 when $\Gamma = \Gamma_1$.

Solution

Case $\mathbf{\Gamma_1}$: It is necessary to parametrize the paths AB and BC before the integral can be evaluated. On AB $z = t + 2i$ for $2 \le t \le 6$, so an increase in t induces a sense on AB from A to B. Differentiation shows that $dz = dt$ on AB. Similarly, on BC $z = 6 + it$ for $2 \le t \le 3$, so an increase in t induces a sense on BC from B to C. Differentiation shows that $dz = idt$ on BC. We have

$$\begin{aligned}\int_{\Gamma_1} f(z)dz &= \int_{AB} f(z)dz + \int_{BC} f(z)dz \\ &= \int_2^6 (t+2i)dt + \int_2^3 (6+it)i\,dt \\ &= \left(\frac{1}{2}t^2 + 2it\right)\Bigg|_{t=2}^{t=6} + \left(-\frac{1}{2}t^2 + 6it\right)\Bigg|_{t=2}^{t=3} = \frac{27}{2} + 14i.\end{aligned}$$

Case $\mathbf{\Gamma_2}$: Elementary coordinate geometry shows that the straight line through AC has the equation

$$y = \frac{3}{2} + \frac{x}{4},$$

so the line segment AC on this line is described by the condition $2 \le x \le 6$. This shows that a general point z on AC has the parametrization $x = t$, $y = \frac{3}{2} + \frac{t}{4}$ with $2 \le t \le 6$, and so

$$z(t) = t + i\left(\frac{3}{2} + \frac{t}{4}\right), \quad \text{for } 2 \le t \le 6.$$

Using this parametrization, an increase in t induces a sense from A to C on AC. Differentiation shows that $dz = (1 + \frac{i}{4})dt$, and so

$$\begin{aligned}\int_{\Gamma_2} f(z)dz &= \int_{AC} f(z)dz = \int_2^6 \left(t + i\left(\frac{3}{2} + \frac{t}{4}\right)\right)\left(1 + \frac{i}{4}\right)dt \\ &= \int_2^6 \left(\frac{15}{16}t - \frac{3}{8}\right)dt + i\int_2^6 \left(\frac{3}{2} + \frac{t}{2}\right)dt = \frac{27}{2} + 14i.\end{aligned}$$

***Case* Γ_3:** As $\Gamma_3 = AB + BC + CA$, $\int_{\Gamma_3} z dz = \int_{\Gamma_1} z dz + \int_{CA} z dz$, but $\int_{\Gamma_1} z dz = \frac{27}{2} + 14i$, and from (6), $\int_{CA} z dz = -\int_{AC} z dz = -\frac{27}{2} - 14i$, so

$$\int_{\Gamma_3} z dz = \frac{27}{2} + 14i - \left(\frac{27}{2} + 14i\right) = 0.$$

To verify Theorem 14.2 for the path Γ_1 we proceed as follows. As $\int_{\Gamma_1} z dz = \frac{27}{2} + 14i$,

$$\left|\int_{\Gamma_1} z dz\right| = \left|\frac{27}{2} + 14i\right| = \frac{1}{2}\sqrt{1513} = 19.45.$$

On AB $z = t + 2i$, so $|z| = (t^2 + 4)^{1/2}$, and this assumes its largest value on AB at B when $t = 6$, so $\max_{AB} |z| = 40^{1/2} = 6.32$. On BC $z = 6 + it$, so $|z| = (t^2 + 36)^{1/2}$, and this assumes its largest value on BC at C when $t = 3$, so $\max_{BC} |z| = 45^{1/2} = 6.71$. These results show that M, the greatest value of $|z|$ on Γ_1, is $M = 6.71$. The length L of path $\Gamma_1 = 4 + 1 = 5$, so $ML = 6.71 \times 5 = 33.55$, which is greater than $|\int_{\Gamma_1} z dz| = 19.45$, so the result of Theorem 14.2 is confirmed. ■

EXAMPLE 14.4 Show that

$$\int_{\Gamma} (z - z_0)^n dz = 0, \quad \text{for } n \neq -1 \text{ a positive or negative integer;}$$

where Γ is a circle of radius R centered on the point $z = z_0$, and integration is performed around Γ in the counterclockwise sense.

Solution It can be seen from Example 14.1 that the contour Γ in Fig. 14.6 can be parametrized by setting $z(t) = z_0 + Re^{it}$, with $0 \leq t \leq 2\pi$.

Using this parametrization, an increase in t, induces a sense of direction around contour Γ in the counterclockwise (positive) direction, and differentiation of $z(t)$ with respect to t shows that on Γ we have $dz = i Re^{it} dt$. Substituting for $z - z_0$ and dz, we obtain

$$\int_{\Gamma} (z - z_0)^n dz = \int_0^{2\pi} R^n e^{int} i Re^{it} dt = i R^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt$$

$$= i R^{n+1} \left(\frac{\exp[i(n+1)t]}{i(n+1)}\right)_{t=0}^{t=2\pi} = 0, \quad \text{provided } n \neq -1. \quad ■$$

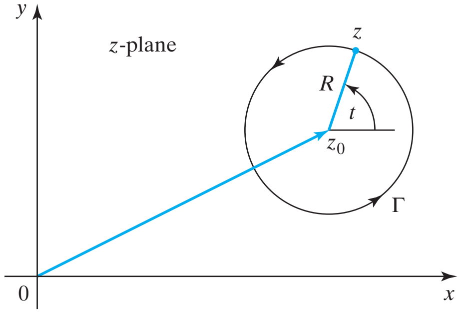


FIGURE 14.6 The circle Γ.

EXAMPLE 14.5 Show that

$$\int_\Gamma \frac{dz}{z-z_0} = 2\pi i,$$

where Γ is the circular contour used in Example 14.4.

Solution Using the parameterization of Example 14.4 we find that the integrand becomes $dz/(z-z_0) = Rie^{it}dt/Re^{it} = i\,dt$, so

$$\int_\Gamma \frac{dz}{z-z_0} = i\int_0^{2\pi} dt = 2\pi i. \quad \blacksquare$$

zeros and poles

The integrands in Examples 14.4 and 14.5 are special cases of functions which possess what are called *zeros* and *poles*. To make matters precise, a function $f(z)$ is said to have a **zero of order *n*** at $z = z_0$ if $n \geq 1$ is an integer and

$$f(z) = (z-z_0)^n g(z), \quad \text{with } g(z_0) \neq 0. \tag{8}$$

Expressed differently, $f(z)$ will have a zero of order n at $z = z_0$ if

$$\lim_{z\to z_0} \frac{f(z)}{(z-z_0)^n} = g(z_0), \quad \text{with } g(z_0) \neq 0.$$

A function $f(z)$ will have a **pole of order *n*** at $z = z_0$ if $n \geq 1$ is an integer and

$$f(z) = \frac{g(z)}{(z-z_0)^n}, \quad \text{with } g(z_0) \neq 0. \tag{9}$$

Expressed differently, $f(z)$ will have a pole of order n at $z = z_0$ if

$$\lim_{z\to z_0} (z-z_0)^n f(z) = g(z_0), \quad \text{with } g(z_0) \neq 0.$$

This shows that when $n \geq 1$ the integrand in Example 14.4 has a zero of order n at $z = z_0$ with $g(z) = 1$, and when $n \leq -1$ a pole of order $|n|$ at $z = z_0$ with $g(z) = 1$. The integrand in Example 14.5 has a pole of order 1, called a **simple pole**, at $z = z_0$ with $g(z) = 1$.

Similarly, the function

$$f(z) = \frac{(z-2)^3}{(z-1)(z+5)^2}$$

has zero of order 3 at $z = 2$, a simple pole at $z = 1$ and pole of order 2 at $z = -5$.

This definition of a pole will be used first in Theorem 14.14, though later the simple poles of functions will be seen to play an essential role in complex integration.

Summary

The positive (counterclockwise) sense of direction around contours was defined and the line integral of a complex function was introduced. The useful *ML* theorem that estimates the magnitude of a complex line integral was derived and two elementary integrals around simple closed loops (contour integrals) were found.

EXERCISES 14.1

1. Given that A, B, and C are the respective points (2, 1), (4, 2), and (5, 4) in the z-plane, find parametric representations of the straight line segments AB and BC with their respective senses from A to B and from B to C.
2. Find parametric representations for the straight line segments AB and BC illustrated in Fig. 14.7, with the senses shown by the arrows.

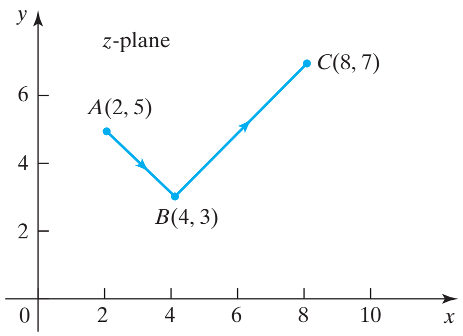


FIGURE 14.7 The straight line segments AB and BC.

3. Find parametric representations for the straight line segments AB and BC illustrated in Fig. 14.8, with the senses shown by the arrows.

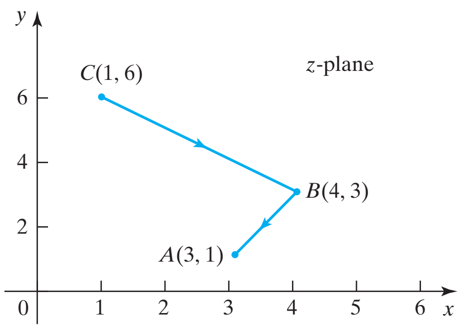


FIGURE 14.8 The line segments AB and BC.

4. Find parametric representations for the straight line segment AB, the circular arc BC, and the straight line segment CD illustrated in Fig. 14.9, with the senses shown by the arrows.
5. Integrate $f(z) = z$ in the positive sense around the square with corners at (1, 1), (2, 1), (2, 2), and (1, 2).
6. Integrate $f(z) = z$ along the consecutive straight line paths from A to B and from B to C, where A, B, and C are the respective points (1, 1), (3, 2), and (5, 4).

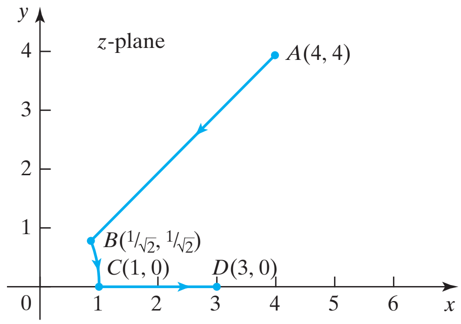


FIGURE 14.9 The straight line segments AB and CD, and the circular arc BC.

7. Integrate $f(z) = z^2 + i$ along the straight line path from point (1, 1) to (1, 4).
8. Integrate $f(z) = iz^2 + 1$ along the straight line path from point (3, 1) to (6, 1).
9. Integrate $f(z) = 2z^2 - 3i$ along the straight line path from point (1, 1) to (4, 1).
10. Integrate $f(z) = z^2 + z$ along the straight line path from point (2, 3) to (5, 6).
11. Represent sinh z in terms of its real and imaginary parts and integrate it along the straight line path from point $(3, \pi)$ to $(6, \pi)$.
12. Represent cosh z in terms of its real and imaginary parts and integrate it along the straight line path from point (1, 2) to (1, 4).
13. Represent sin z in terms of its real and imaginary parts and integrate it along the straight line path from point $(2, \pi)$ to $(3, \pi)$.
14. Represent cos z in terms of its real and imaginary parts and integrate it along the straight line path from point $(1, 4\pi)$ to $(1, 6\pi)$.
15. Represent cosh $2z$ in terms of its real and imaginary parts and integrate it along the straight line path from point (0, 0) to (4, 2).
16. Represent sin z in terms of its real and imaginary parts and integrate it along the straight line path from point (0, 0) to (2, 4).
17. Integrate e^z along the straight line path from the point (0, 0) to $(4, \pi/4)$.
18. Set $f(z) = \bar{z}$, and let the corners A, B, C, and D of a square be located at the respective points $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$. Integrate $f(z)$ first along the consecutive paths from A to B and from B to C, and then along the consecutive paths from A to D and from D to C, and hence show that the value of the

integral of the nonanalytic function $\bar{z}$ from A to C depends on the choice of path joining A to C.

19. Integrate $f(z) = 1/(z-1)$ in the negative sense around the semicircle with the equation $|z-1| = 1$.

20. Integrate the function $f(z) = z\bar{z}$ around the circular arc $|z-2| = 3$ in the positive sense between the points $(2, 3)$ and $(5, 0)$.

21. Show that $\int_\Gamma \frac{1}{z+i} dz = 0$, when integration is performed in the either the positive or the negative sense around the circle Γ given by $|z-2| = 2$.

22. Let A, B, and C be the respective points $(0, 0)$, $(1, 0)$, and $(1, 1)$, and let $f(z) = z\bar{z}$. Integrate $f(z)$ along the consecutive straight line segments AB and BC, and then along the straight line segment AC, and hence show that the value of the integral of this nonanalytic function from A to C depends on the path joining the two points.

14.2 Contours, the Cauchy–Goursat Theorem, and Contour Integrals

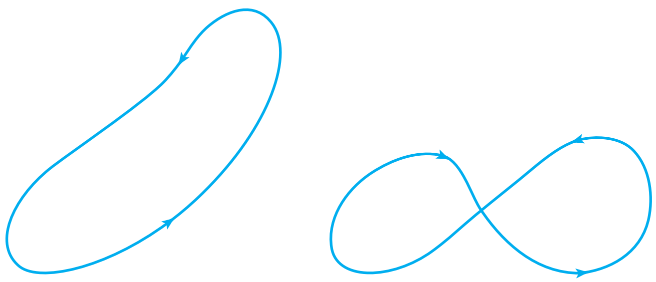
The definition of a complex integral of a single-valued analytic function $f(z)$ along a path introduced in Section 14.1 was for paths that were finite in length, did not intersect themselves, and had end points that were distinct. To make further progress with complex integrals it is necessary to consider integrating along general paths in the form of closed loops that are continuous, piecewise smooth, and do not intersect themselves. In Section 14.1, closed paths of this type were called **contours**, though they are also often called **simple closed curves** or **Jordan curves**. A typical example of a simple closed curve is shown in Fig. 14.10a, and the self-intersecting figure-eight-shaped curve in Fig. 14.10b is a nonsimple closed curve.

contours and simple closed curves

Before examining contour integrals in more detail, it is necessary to introduce the notion of a *simply connected* domain in which all contour integrals are to be evaluated. A domain D is called **simply connected** if the interior points of all possible simple closed curves in D belong to D. This means that a simply connected domain is one from which no points, curves, or areas are missing. A domain D that does not satisfy this condition is said to be **multiply connected**.

simply and multiply connected domains

An example of a simply connected domain is shown in Fig. 14.11a, and typical multiply connected domains are shown in Figs. 14.11b and c. The annular domain in Fig. 14.11b is a simple example of a multiply connected domain, and it is made multiply connected by the removal from D of the points in the disc in the center that leaves a "hole" in D. Domains containing only one "hole" are said to be **doubly**



(a) (b)

FIGURE 14.10 (a) A simple closed curve. (b) A nonsimple closed curve.

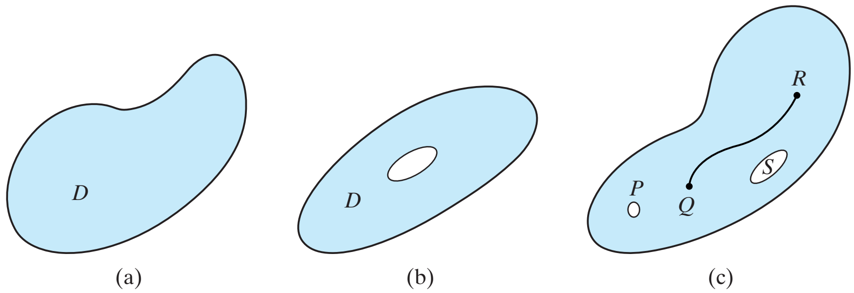


FIGURE 14.11 (a) Simply, (b) doubly, and (c) multiply connected domains.

connected. The domain in Fig. 14.11c is *multiply connected* because the point at P is missing, as are the points along the cut QR and the points in the area (hole) S.

Another way of defining a simply connected domain D is by saying it is one with the property that *every* simple closed curve connecting any two points of D can *always* be collapsed onto an arc in D that joins the two points. This definition is illustrated in Fig. 14.12a, from which it can be seen that for any two points A and B in D, all simple closed curves Γ connecting A and B can always be collapsed onto a dashed arc like the one shown joining the two points. Domain D in Fig. 14.12b is multiply connected. The reason for this can be seen by examining the curves Γ_1 and Γ_2. The simple closed curve Γ_1 joining two points A and B in D lies entirely to the side of all holes in D, and so can be collapsed onto an arc in D joining the points A and B, but this is not possible for a simple closed curve such as Γ_2 that encloses one or more of the holes in D, because the boundaries of the holes act as barriers that stop its collapse onto an arc.

In future the notation $\int_\Gamma f(z)dz$, already used to denote the line integral of a single-valued analytic function $f(z)$ along a path Γ, will be taken to include **contour integrals** around a simple closed curve Γ.

The fundamental theorem governing contour integrals is the **Cauchy–Goursat** theorem, which can be stated as follows.

THEOREM 14.3

a fundamental theorem

Cauchy–Goursat Theorem Let f be a single-valued analytic function in a simply connected domain D. Then if Γ is any simple closed curve of finite length lying

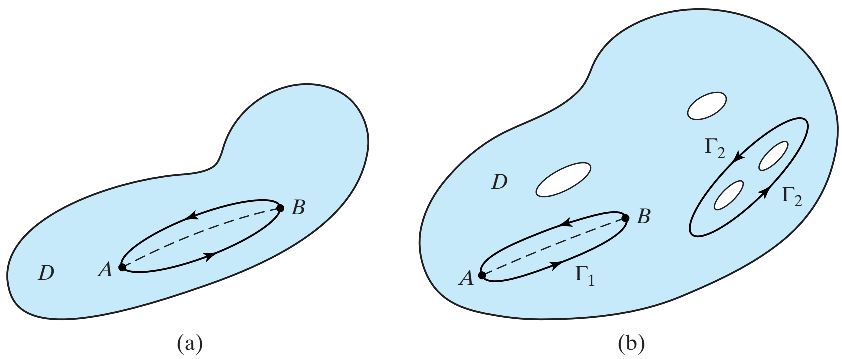


FIGURE 14.12 Illustration of the alternative definition of simply and multiply connected domains.

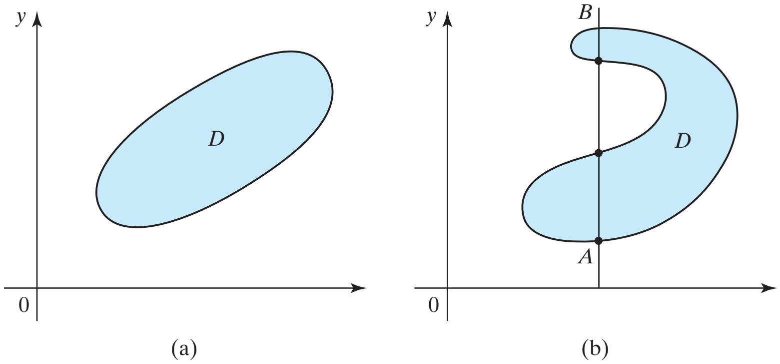


FIGURE 14.13 Standard and nonstandard domain.

entirely within D,

$$\int_{\Gamma} f(z)dz = 0.$$

Proof This is the most general statement of the Cauchy–Goursat theorem that is necessary for practical purposes. We now prove it in a weaker form by requiring that in addition to f being single-valued and analytic, its derivative $f'(z)$ must be continuous in D and the contour Γ must be one for which lines passing through the interior of Γ drawn parallel to the real and imaginary axes only intersect Γ twice. Areas bounded by such closed curves Γ are called **standard domains**. A typical standard domain is shown in Fig. 14.13a, and a nonstandard one is shown in Fig. 14.13b, where lines such as AB are seen to intersect D four times.

standard and nonstandard domains

Under the stated conditions, the proof can be based on Green's theorem in the plane, which takes the form

$$\int_{\Gamma} Pdx + Qdy = \int_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

where the domain D inside Γ is a simple domain and P, Q, $\partial Q/\partial x$, and $\partial P/\partial y$ are continuous in D and on Γ. If $f(z) = u + iv$, then $f'(z) = \partial u/\partial x + i\partial v/\partial x = \partial v/\partial y - i\partial u/\partial y$, so the assumption that $f'(z)$ is continuous implies the continuity of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$, and through them the continuity of u and v.

Applying Green's theorem to $\int_{\Gamma} f(z)dz$, we have

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (udx - vdy) + i \int_{\Gamma} (vdx + udy)$$
$$= \int_{D} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i \int_{D} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy.$$

However, from the Cauchy–Riemann equations $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, so each integrand vanishes and we obtain the statement of the theorem

$$\int_{\Gamma} f(z)dz = 0.$$

The form of proof given here is the one due to Cauchy. The removal of the

requirements that $f'(z)$ be continuous and D be a standard domain that were necessary in the above proof allows the theorem to be used under very general circumstances. It means, for example, that the theorem remains true when domains such as the one in Fig. 14.13b arise, and also that instead of the contour Γ being smooth, it can be formed from piecewise smooth arcs joined end to end to make a simple closed curve such as a semicircle or a rectangle. The generalization of the theorem is due to Goursat, though the details of its proof will not be given here. ■

EXAMPLE 14.6

The functions z^n with n a positive integer, $\sin z$, $\cos z$, e^z, $\sinh z$, and $\cosh z$ are analytic and single valued throughout the complex plane (they are *entire* functions), so for any simple contour Γ,

$$\int_\Gamma z^n dz = 0, \quad \int_\Gamma \sin z dz = 0, \quad \int_\Gamma \cos z dz = 0, \quad \int_\Gamma e^z dz = 0,$$

$$\int_\Gamma \sinh z dz = 0, \quad \int_\Gamma \cosh z dz = 0.$$ ■

EXAMPLE 14.7

The function $\sec z = 1/\cos z$ is analytic and single valued throughout the z-plane except at the zeros of $\cos z$ that are located at $z = (2n+1)\pi/2$, for $n = 0, \pm 1, \pm 2, \ldots$. Thus, $\int_\Gamma \sec z dz = 0$ for every contour Γ that neither contains zeros of $\cos z$ nor passes through any of its zeros. ■

An immediate consequence of the Cauchy–Goursat theorem is that if the contour Γ in D is **deformed** into some other contour Γ_1 that is also in D, the statement in the theorem remains unchanged. When this happens the contours Γ and Γ_1 are said to be **equivalent contours**.

Examples of two equivalent contours are shown in Fig. 14.14a, and the usefulness of this result is such that we record it in the form of a theorem.

THEOREM 14.4

a suitable deformation of a contour does not change the value of a contour integral

Deformation of contours Let f be a single-valued analytic function in a simply connected domain D, and let Γ_1 and Γ_2 be any two simple closed contours in D.

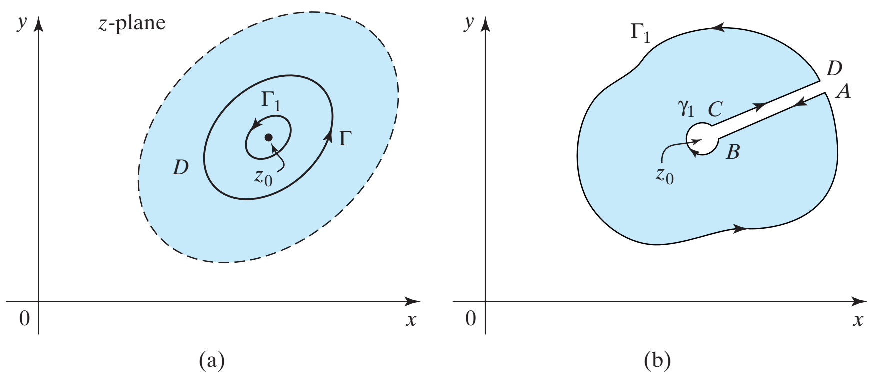


FIGURE 14.14 (a) Equivalent contours. (b) A contour that excludes a simple pole at z_0.

Then Γ_1 and Γ_2 are equivalent in the sense that

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

If, however f has a simple pole at a point $z = z_0$ inside both Γ_1 and Γ_2, then

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz = 2\pi i \lim_{z \to z_0} [(z - z_0) f(z)].$$

Proof The first result has already been established, so it only remains to prove the second one. Consider Fig. 14.14b, and let there be a simple pole at $z = z_0$. Enclose the pole in a small circle γ_1 of radius r, and join the circle to the contour Γ_1 by two parallel straight lines AB and CD that are arbitrarily close together. Then, in the domain bounded by Γ_1, AB, γ_1, and CD as indicated by the arrows in Fig. 14.14b, the function f is analytic because the pole has been excluded.

Applying the Cauchy–Goursat theorem and integrating around this contour gives

$$\int_{\Gamma_{DA}} f(z)dz + \int_{AB} f(z)dz + \int_{-\gamma_1} f(z)dz + \int_{CD} f(z)dz = 0,$$

where $-\gamma_1$ indicates that integration around the circle γ_1 is in the clockwise sense.

If the radius r of circle γ_1 is now allowed to tend to zero, the second and fourth integrals vanish, because f is continuous across the lines AB and CD and f is integrated in opposite directions along each of these lines. Reversing the sense of integration around γ_1 and compensating by changing the sign of the integral, we arrive for $r \to 0$ at the result

$$\int_{\Gamma_1} f(z)dz = \lim_{r \to 0} \int_{\gamma_1} f(z)dz.$$

By definition, if f has a simple pole at $z = z_0$, then $f(z) = g(z)/(z - z_0)$ with $g(z_0) \neq 0$. So, integrating around γ_1 on which $z = z_0 + re^{i\theta}$ with $0 \leq \theta \leq 2\pi$, and using the fact that $dz = ire^{i\theta} d\theta$, gives

$$\int_{\Gamma_1} f(z)dz = \lim_{r \to 0} \int_0^{2\pi} \frac{g(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta = 2\pi i g(z_0).$$

The same result would be obtained using any other contour in D that contains z_0, so the second result is proved. ■

EXAMPLE 14.8

Find $\int_\Gamma \frac{3}{z+i} dz$, with Γ any square of side 4 with its center at the origin.

Solution The square Γ contains $z = -i$, which is a simple pole of the integrand, so deforming Γ into any circle centered on $z = -i$ and integrating around Γ in the positive sense using the result of Example 14.5 gives

$$\int_\Gamma \frac{3}{z+i} dz = 6\pi i.$$

■

EXAMPLE 14.9

Find $\int_\Gamma (\frac{4}{z-1} - \frac{5}{z+4})dz$, where Γ is the circle $|z| = 2$.

Solution The point $z = -4$ lies outside $|z| = 2$, so the Cauchy–Goursat theorem shows that the second term in the integrand contributes nothing to the integral. Deforming Γ into any circle centered on $z = 1$ that does not contain the point $z = -4$, and integrating around it in the positive sense using the result of Example 14.5, gives

$$\int_\Gamma \left(\frac{4}{z-1} - \frac{5}{z+4}\right) dz = 8\pi i - 0 = 8\pi i.$$ ■

EXAMPLE 14.10

simplifying integration by using partial fractions

Find

$$\int_\Gamma \frac{2z-3}{z^3 - 3z^2 + 4} dz$$

by integrating in the positive sense around Γ when (a) Γ is the circle $|z| = 3/2$ and (b) Γ is the circle $|z - 3| = 2$.

Solution A partial fraction decomposition of the integrand gives

$$\frac{2z-3}{z^3 - 3z^2 + 4} = \frac{5}{9}\frac{1}{z-2} - \frac{5}{9}\frac{1}{z+1} + \frac{1}{3}\frac{1}{(z-2)^2},$$

so

$$\int_\Gamma \frac{2z-3}{z^3 - 3z^2 + 4} dz = \frac{5}{9}\int_\Gamma \frac{dz}{z-2} - \frac{5}{9}\int_\Gamma \frac{dz}{z+1} + \frac{1}{3}\int_\Gamma \frac{dz}{(z-2)^2}.$$

(a) The functions $1/(z-2)$ and $1/(z-2)^2$ are analytic in and on the circle $|z| = 3/2$, so by the Cauchy–Goursat theorem the first and last integrals on the right vanish. The contour Γ is not convenient for the evaluation of the second integral on the right, so we deform the circle $|z| = 3/2$ into the circle $|z + 1| = 1$ centered on $z = -1$ and use the result of Example 14.5 to obtain

$$\int_\Gamma \frac{dz}{z+1} = 2\pi i.$$

Combining these results gives

$$\int_\Gamma \frac{2z-3}{z^3 - 3z^2 + 4} dz = -\frac{5}{9}\int_\Gamma \frac{dz}{z+1} = -\frac{10\pi i}{9}.$$

(b) The function $1/(z+1)$ is analytic in and on the circle $|z-3| = 2$, so by the Cauchy–Goursat theorem the second integral on the right vanishes. Again the contour Γ is not convenient when determining the other two contour integrals, so deforming the circle $|z - 3| = 2$ into the circle $|z - 2| = 1$ and using the results of Examples 14.4 and 14.5 gives

$$\int_\Gamma \frac{dz}{z-2} = 2\pi i, \quad \text{and} \quad \int_\Gamma \frac{dz}{(z-2)^2} = 0.$$

Combining these results we find that

$$\int_\Gamma \frac{2z-3}{z^3 - 3z^2 + 4} dz = \frac{5}{9}\int_\Gamma \frac{dz}{z-2} = \frac{10\pi i}{9}.$$ ■

Let f be a single-valued analytic function in some domain D in which two distinct points z_1 and z_2 are connected by two paths in D that form the simple contour Γ shown as $APBQA$ in Fig. 14.15a.

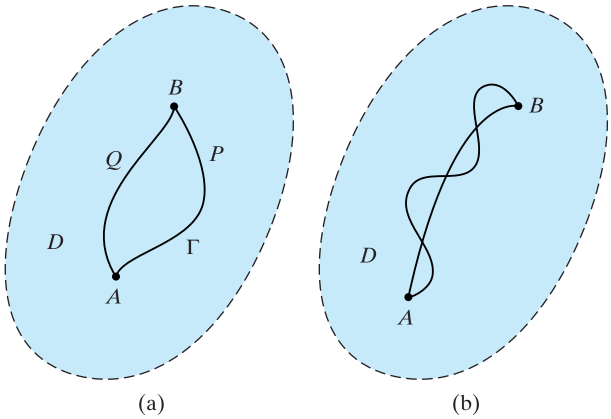


FIGURE 14.15 (a) Two paths forming a simple contour Γ. (b) Two paths forming loops.

Using the Cauchy–Goursat theorem and dividing Γ into the two parts APB and BQA allow us to write

$$\int_{\Gamma} f(z)dz = \int_{APB} f(z)dz + \int_{BQA} f(z)dz = 0.$$

Reversing the direction of integration along BQA, and compensating by changing the sign of the integral, shows the preceding result to be equivalent to

$$\int_{APB} f(z)dz = \int_{AQB} f(z)dz. \tag{10}$$

By Theorem 14.4 the contour Γ in D through z_1 and z_2 can be deformed into any other equivalent contour in D through the two points, showing that the integral of $f(z)$ from z_1 to z_2 is independent of the path joining z_1 to z_2. The result remains true if the paths intersect finitely many times forming n loops, as shown in Fig. 14.15b. In this case the result is established by applying the preceding result to each loop in succession.

antiderivative, or indefinite integral

As in the real variable calculus, a differentiable function $F(z)$ such that $F'(z) = f(z)$ is called an **antiderivative** of $f(z)$, or an **indefinite integral**, and written

$$\int f(z)dz. \tag{11}$$

To simplify the calculation of line integrals of analytic functions, we now consider the integral of a single-valued analytic (and so *continuous*) function $f(z)$ from a fixed point z_0 in D to some other point z in D along any path in D. The result can be written

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta, \tag{12}$$

where $F(z)$ is a function of the upper limit of integration z, and no path need be specified because the integral is independent of the path joining z_0 to z_1 in D.

We wish to show that $F'(z) = f(z)$, so let us consider the difference quotient

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta,$$

where Δz is a small increment in z.

As any path in D between z and $z + \Delta z$ can be used, we take it to be the straight line segment joining these two points. Then, as $\int_z^{z+\Delta z} d\zeta = \Delta z$, we can multiply this result by $f(z)/\Delta z$ and use the fact that $f(z)$ is not involved in the integration to write $f(z)$ as

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)d\zeta.$$

This result allows the difference quotient to be written

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]d\zeta.$$

Taking the modulus of this expression and using the fundamental integral inequality in Theorem 14.1, we obtain

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \leq \frac{1}{|\Delta z|}\int_{z}^{z+\Delta z} |f(\zeta) - f(z)||d\zeta|,$$

but $f(z)$ is a continuous function of z, so for any arbitrary small number $\varepsilon > 0$ we can always find a number $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \varepsilon, \quad \text{when } |z - \zeta| < \delta.$$

Then, as ζ lies on the straight line segment joining z and $z + \Delta z$, we have $|z - \zeta| \leq |\Delta z|$, showing that the preceding result is true if $\delta < |\Delta z|$.

It now follows that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \leq \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon,$$

so in the limit as $\Delta z \to 0$ this shows that

$$\lim_{\Delta z\to 0}\left(\frac{F(z+\Delta z) - F(z)}{\Delta z}\right) = F'(z) = f(z). \tag{13}$$

As $F(z)$ has been shown to be differentiable, we have also proved the very important result that the derivative of an analytic function is itself an analytic function.

We now show how definite integrals can be evaluated. Let $F(z)$ and $G(z)$ be any two different antiderivatives of $f(z)$. Then setting $\Phi(z) = F(z) - G(z) = u + iv$, we have

$$\Phi'(z) = F'(z) - G'(z) = 0, \quad \text{for all } z \text{ in } D.$$

When this result is used with the Cauchy–Riemann equations, it shows that $\Phi(z) =$ constant, so all antiderivatives of $f(z)$ can only differ one from the other by a complex constant C, allowing us to write

$$F(z) = G(z) + C.$$

If z and z^* are any two points in D where f is defined, the antiderivative $G(z)$ of $f(z)$ can be written

$$G(z) = \int_{z^*}^{z} f(\zeta)d\zeta, \tag{14}$$

so the most general antiderivative of $f(z)$ becomes

$$F(z) = \int_{z^*}^{z} f(\zeta)d\zeta + C. \tag{15}$$

The **definite integral** $\int_{z_0}^{z_1} f(\zeta)d\zeta$ can be written

$$\int_{z^*}^{z_1} f(\zeta)d\zeta = \int_{z^*}^{z_1} f(\zeta)d\zeta - \int_{z^*}^{z_0} f(\zeta)d\zeta,$$

and after elimination of the arbitrary constant C we find that

$$\int_{z_0}^{z_1} f(\zeta)d\zeta = F(z_1) - F(z_0). \tag{16}$$

In complex analysis, this last result is the analogue of the fundamental theorem of integral calculus for real functions. We have proved the following important and useful theorem.

THEOREM 14.5

Independence of path—definite integrals Let $f(z)$ be a single-valued analytic function in some domain D to which belong the two distinct points z_1 and z_2. Then if $F(z) = \int f(z)dz$ is an antiderivative of f, the line integral of f along any path in D joining z_1 to z_2 is independent of the path, and

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1).$$ ■

EXAMPLE 14.11

Find the integral of z^2 from $z_1 = 1 + i$ to $z_2 = 3 + 4i$.

Solution The function $f(z) = z^2$ is single valued and analytic in the finite z-plane, and an antiderivative of $f(z)$ is $z^3/3$, so Theorem 14.5 can be applied and gives

$$\int_{1+i}^{3+4i} z^2 dz = \left(\frac{z^3}{3}\right)\Bigg|_{1+i}^{3+4i} = \frac{1}{3}[(3+4i)^3 - (1+i)^3] = -\frac{115}{3} + 14i.$$ ■

Consider a function $f(z)$ that is analytic and single valued inside the multiply connected domain D with outer boundary Γ shown in Fig. 14.16a. The domain D can be made simply connected by inserting the n cuts $C_1, C_2, \ldots, C_n$ shown in Fig. 14.16b, and taking as the new boundary the one formed by Γ, the internal boundaries $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, and the cuts $C_1, C_2, \ldots, C_n$. In this way, as the contour is traversed in the positive sense indicated by the arrows in Fig. 14.16b, the modified domain always lies to the left and is simply connected.

The next theorem makes use of cuts to extend the Cauchy–Goursat theorem for analytic functions to multiply connected domains.

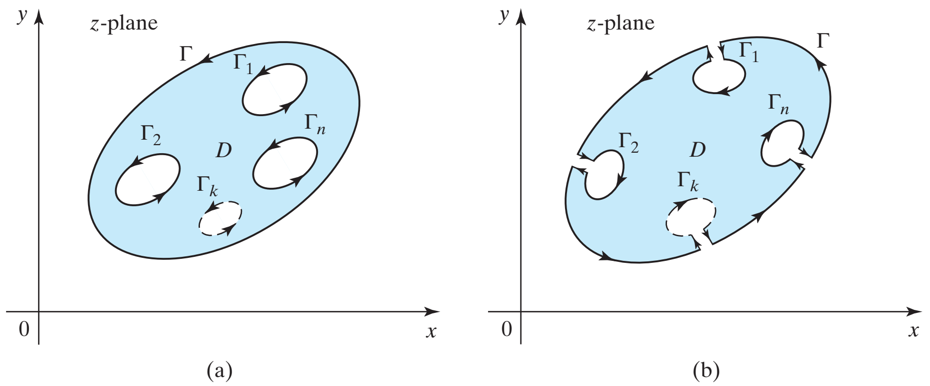


FIGURE 14.16 Cuts used to make a multiply connected domain simply connected.

THEOREM 14.6

integration in multiply connected domains

Extended Cauchy–Goursat theorem Let $f(z)$ be a single-valued analytic function in a possibly multiply connected domain D bounded externally by a simple contour Γ, and internally by the simple contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, as shown in Fig. 14.17, and let each of the $n+1$ contours be traversed in the positive sense. Then

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \cdots + \int_{\Gamma_n} f(z)dz.$$

Proof Make the cuts indicated in Fig. 14.18, and integrate around the resulting composite contour using the Cauchy–Goursat theorem to obtain

$$\begin{aligned}&\int_{c_1^+} f(z)dz + \int_{\Gamma_1} f(z)dz + \int_{c_1^-} f(z)dz + \int_{\Gamma(P_1^- P_2^+)} f(z)dz + \int_{c_2^+} f(z)dz\\ &+ \int_{\Gamma_2} f(z)dz + \int_{c_2^-} f(z)dz + \cdots + \int_{c_n^+} f(z)dz + \int_{\Gamma_n} f(z)dz + \int_{c_n^-} f(z)dz\\ &+ \int_{\Gamma(P_n^- P_1^+)} f(z)dz = 0.\end{aligned}$$

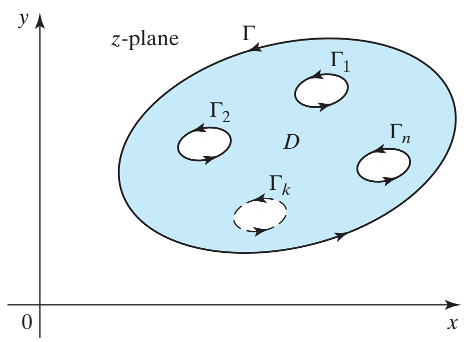


FIGURE 14.17 The multiply connected domain D.

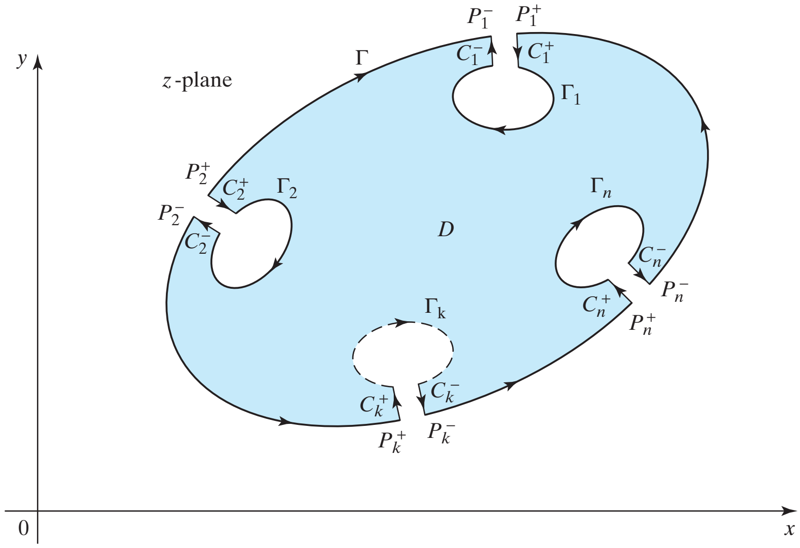


FIGURE 14.18 Composite contour for integration.

As f is analytic in D, and C_i^+ and C_i^- are opposite sides of the cut C_i, the function f is continuous across the cut. The paths C_i^+ and C_i^- are traversed in opposite directions, so the integrals along opposite sides of the cut cancel, leading to the result

$$\int_{c_i^+} f(z)dz + \int_{c_i^-} f(z)dz = 0, \quad \text{for } i = 1, 2, \ldots, n.$$

Adding the integrals around the successive segments of Γ, using the fact that $f(z)$ is continuous on Γ, cancelling the integrals along opposite sides of each cut, and denoting integration around Γ_i in the clockwise (negative) sense by Γ_{i-} reduces the preceding result to

$$\int_{\Gamma} f(z)dz + \int_{\Gamma_{1-}} f(z)dz + \int_{\Gamma_{2-}} f(z)dz + \cdots + \int_{\Gamma_{n-}} f(z)dz = 0.$$

The direction of integration around the internal contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is negative (clockwise), so reversing their directions to give them a positive orientation, introducing corresponding changes of sign in the integrals, and rearranging terms, we arrive at the result

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \cdots + \int_{\Gamma_n} f(z)dz,$$

and the theorem is proved. ■

EXAMPLE 14.12 Find the integral of $f(z) = (4z^2 + 11z - 3)/(z^3 + 2z^2 - z - 2)$ around the contour Γ shown in Fig. 14.19 with the direction of integration around the connected contours A, B, and C shown by the arrows.

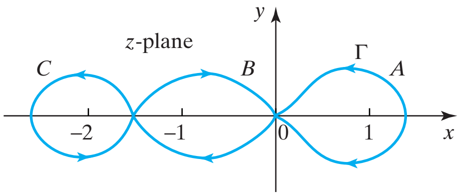


FIGURE 14.19 Connected contours A, B, and C forming Γ.

Solution Integrating around Γ we have

$$\int_\Gamma f(z)dz = \int_A f(z)dz + \int_B f(z)dz + \int_C f(z)dz,$$

and a partial fraction expansion of $f(z)$ gives the representation

$$f(z) = \frac{2}{z-1} + \frac{5}{z+1} - \frac{3}{z+2}.$$

Inside and on contour A the functions $1/(z+1)$ and $1/(z+2)$ are analytic; inside and on contour B the functions $1/(z-1)$ and $1/(z+2)$ are analytic; and inside and on contour C the functions $1/(z-1)$ and $1/(z+1)$ are analytic. In addition, we must take account of the fact that integration around A is in the *positive* sense, integration around B is in the *negative* sense, and integration around C is in the *positive* sense. Deforming contours A, B, and C into the respective circles $|z-1| = 1$, $|z+1| = 1/2$, and $|z+2| = 1/2$ and using the Cauchy–Goursat theorem with the result of Example 14.5, we find that integration around contour A in the *positive* sense gives

$$\int_A f(z)dz = 2\int_{|z-1|=1} \frac{1}{z-1}dz = 2\cdot 2\pi i = 4\pi i,$$

integration around contour B in the *negative* sense gives

$$\int_B f(z)dz = -5\int_{|z+1|=1/2} \frac{1}{z+1}dz = -5\cdot 2\pi i = -10\pi i,$$

and integration around contour C in the *positive* sense gives

$$\int_C f(z)dz = -3\int_{|z+2|=1/2} \frac{1}{z+2}dz = -3\cdot 2\pi i = -6\pi i.$$

Adding these results to find the integral around Γ we obtain

$$\int_\Gamma f(z)dz = 4\pi i - 10\pi i - 6\pi i = -12\pi i.$$ ■

integrands involving quotients of trigonometric functions

By setting $z = e^{i\theta}$, expressing $\sin\theta$ and $\cos\theta$ in terms of z, and integrating around the unit circle Γ given by $|z| = 1$, the Cauchy–Goursat theorem can be used to evaluate trigonometric integrals of the form

$$\int_0^{2\pi} \frac{a\cos\theta + b\sin\theta}{c + d\cos\theta + e\sin\theta}d\theta, \qquad (17)$$

where a, b, c, d, and e are real numbers.

The expressions for $\sin\theta$ and $\cos\theta$ in terms of z follow by adding and subtracting

$$z = \cos\theta + i\sin\theta \quad \text{and} \quad 1/z = \cos\theta - i\sin\theta$$

to obtain

$$\sin\theta = \frac{1}{2i}\left(\frac{z^2-1}{z}\right), \quad \cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right), \tag{18}$$

and differentiating the result $z = e^{i\theta}$ to obtain $dz = ie^{i\theta}d\theta$, from which it follows that

$$d\theta = \frac{1}{iz}dz. \tag{19}$$

EXAMPLE 14.13

Find

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}, \quad \text{where } a \text{ and } b \text{ are real numbers such that } |a/b| > 1.$$

Solution The condition $|a/b| > 1$ is necessary to prevent the integrand becoming unbounded in the interval of integration.

Substituting for $d\theta$ and $\sin\theta$ in the integral, we find that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b}\int_\Gamma \frac{dz}{z^2 + 2i(a/b)z - 1},$$

where Γ is the unit circle, and integration around Γ is in the positive sense.

As $|a/b| > 1$, the roots of the denominator $z^2 + 2i(a/b)z - 1 = 0$ can be written

$$\alpha = \frac{i}{b}(-a + \sqrt{a^2 - b^2}) \quad \text{and} \quad \beta = \frac{i}{b}(-a - \sqrt{a^2 - b^2}),$$

where the positive square root is taken. Then, as $|\alpha| < 1$, the point $z = \alpha$ lies inside Γ, and as $|\beta| > 1$, the point $z = \beta$ lies outside Γ. In terms of α and β the denominator can be written

$$z^2 + 2i(a/b)z - 1 = (z - \alpha)(z - \beta),$$

so when expressed in terms of z and the contour Γ, the integral becomes

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b}\int_\Gamma \frac{dz}{(z-\alpha)(z-\beta)}.$$

A partial fraction expansion of the integrand on the right gives

$$\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{(\alpha-\beta)}\left[\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right],$$

showing that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b(\alpha-\beta)}\int_\Gamma \frac{dz}{z-\alpha} - \frac{2}{b(\alpha-\beta)}\int_\Gamma \frac{dz}{z-\beta}.$$

As only $z = \alpha$ lies inside Γ, it follows from the Cauchy–Goursat theorem and Example 14.5 that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b(\alpha-\beta)} \cdot 2\pi i - 0 = \frac{4\pi i}{b(\alpha-\beta)},$$

so as $b(\alpha - \beta) = 2i\sqrt{a^2 - b^2}$ this simplifies to

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \qquad \text{for } |a/b| > 1.$$

■

Summary

Simply and multiply connected domains were introduced, and the fundamental Cauchy–Goursat theorem of complex analysis for a function in a simply connected domain was proved using Green's theorem. Conditions under which contours can be deformed into more convenient shapes were given, and then used to evaluate some simple contour integrals in terms of two elementary results obtained earlier using circular contours. The Cauchy–Goursat theorem was extended to include multiply connected domains, and some simple definite integrals involving quotients of trigonometric functions were obtained.

EXERCISES 14.2

In Exercises 1 through 4 find $\int_{z_1}^{z_2} f(z)dz$ by parametrizing the given path and using the result to integrate $f(z)$ along Γ from z_1 to z_2. State when Theorem 14.5 can be used to evaluate the integral and, when appropriate, use it to check the result.

1. $f(z) = \sinh z$, and the path Γ is the straight line segment joining the points $z_1 = 1$ and $z_2 = i$.

2. $f(z) = e^{3z}$, and the path Γ is the circular arc $|z - 1| = 1$ joining the points $z_1 = 0$ and $z_2 = 1 + i$.

3. $f(z) = z + \text{Im}\{z\}$, and the path Γ is formed by the straight line segment from $z_1 = 1 + i$ to the point $z^* = 2 + i$ and the straight line segment from the point $z^* = 2 + i$ to $z_2 = 2 + 2i$.

4. $f(z) = 2 + \bar{z}$, and the path Γ is the straight line segment from the point $z_1 = 3i$ to the point $z_2 = 3 + 6i$.

In Exercises 5 through 8 find the integral $\int_\Gamma f(z)dz$, where Γ is the unit circle $|z| = 1$ and integration around Γ is taken in the positive sense, using the Cauchy–Goursat theorem whenever it is appropriate.

5. $f(z) = \tanh z$.

6. $f(z) = (z - 3)^2 + \text{Im}\,\{z\}$.

7. $f(z) = z + \bar{z}^2$.

8. $f(z) = e^z/(z^2 - 2)$.

9. What conditions must be satisfied by a contour Γ in order that $\int_\Gamma f(z)dz = 0$, given that (a) $f(z) = \sin z/(z^2 + 1)$, (b) $f(z) = \csc z$, (c) $f(z) = \text{sech}\, z$, and (d) $f(z) = \coth z$?

10. Find

$$\int_\Gamma \frac{z + 1}{z^2 - 3z + 2} dz$$

where Γ is the contour $ABCADEA$ shown in Fig. 14.20, with integration in the direction indicated by the arrows.

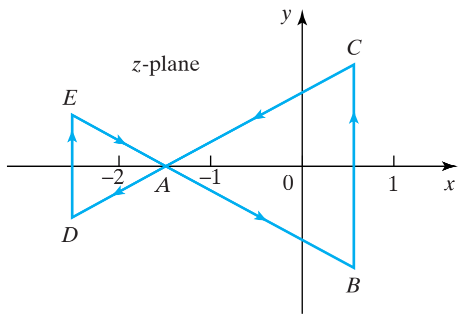


FIGURE 14.20

In Exercises 11 through 17 use analysis to find the integral of $f(z)$ when it is integrated around the given contour Γ in the positive sense. Verify the result by using computer algebra and the substitution $z = z_0 + Re^{i\theta}$, $dz = i\,Re^{i\theta}d\theta$, with $0 \le \theta \le 2\pi$, when Γ is the circle $|z - z_0| = R$.

11. $f(z) = \dfrac{z + 5}{z^2 + 3z - 4}$ with Γ (a) the circle $|z - i| = 2$, and (b) the circle $|z + 3| = 2$.

12. $f(z) = \dfrac{3 - 4z}{z^2 + 5z + 6}$ with Γ (a) the circle $|z| = 5/2$, and (b) the rectangle with its corners at the points $(-7/2, -1)$, $(-5/2, -1)$, $(-5/2, 1)$, and $(-7/2, 1)$.

13. $f(z) = \dfrac{2 - 7z}{z^2 + 3z}$ with Γ (a) the circle $|z + i| = 2$, and (b) the circle $|z - 2| = 4$.

14. $f(z) = \dfrac{3z - 2}{(z + 2)^2}$ with Γ the circle $|z - 3| = 2$.

15. $f(z) = \dfrac{z^2 + 2z}{z^2 - 2z + 1}$ with Γ the circle $|z - 2| = 3$.

16. $f(z) = \dfrac{z + 4}{z^3 + 6z^2 + 9z}$ with Γ (a) the circle $|z + 4| = 2$, and (b) the square with its corners at the points $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$.

17. $f(z) = \dfrac{2z-1}{(z+1)^3}$ with Γ the triangle with its vertices at the points $(-2, -1)$, $(0, -1)$ and $(1, 1)$.

Establish the results of Exercises 18 through 20 by using the method of Example 14.13.

18. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

for a and b real numbers such that $|a/b| > 1$.

19. Show that

$$\int_0^{2\pi} \frac{\cos\theta}{2 + \cos\theta}\, d\theta = 2\pi - \frac{4\pi}{\sqrt{3}}.$$

20. Show that

$$\int_0^{2\pi} \frac{\sin\theta}{3 + \sin\theta}\, d\theta = 2\pi - \frac{3\pi}{\sqrt{2}}.$$

14.3 The Cauchy Integral Formulas

Two consequences of the Cauchy–Goursat theorem are the **Cauchy integral formula** and the **Cauchy integral formula for derivatives** for a function $f(z)$ that is analytic and single valued in some domain D. These results are of fundamental importance in complex analysis and the first of these formulas can be stated as follows.

THEOREM 14.7

The Cauchy integral formula Let $f(z)$ be a single-valued analytic function in a simply connected domain D containing a contour Γ in the form of a simple closed curve. Then for every point z_0 inside Γ,

expressing $f(z_0)$ as an integral

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\, dz,$$

where integration around Γ is in the positive sense.

Proof Let z_0 be any point inside the domain D shown in Fig. 14.21, and let the contour Γ containing z_0 lie inside D. Enclose z_0 by an equivalent circular contour C of arbitrarily small radius ρ.

Let us consider the function $\varphi(z)$ defined as

$$\varphi(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad \text{for } z \neq z_0,$$

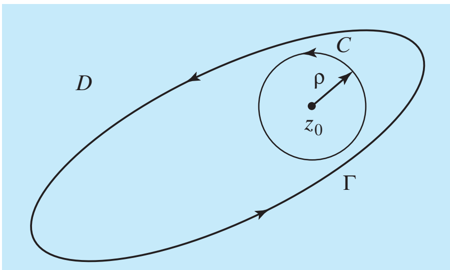


FIGURE 14.21 The equivalent contours Γ and C.

and for later use notice that

$$\lim_{z \to z_0} \varphi(z) = f'(z_0).$$

After deforming the contour Γ into an equivalent circular contour C of radius ρ with its center at z_0, we can write

$$\int_\Gamma \varphi(z)dz = \int_C \varphi(z)dz,$$

where from Example 14.5 it can be seen that the integral around C is independent of the radius ρ. The function $\varphi(z)$ is undefined at $z = z_0$, so if we define it to be $f'(z_0)$ the function $\varphi(z)$ will be continuous throughout D. This result, in turn, implies that the modulus of $\varphi(z)$ must be bounded in D, so we have $|\varphi(z)| \leq M$ for some fixed M and all z in D.

It then follows from Theorem 14.2 that as the circumference of C is $2\pi\rho$,

$$\left| \int_C \varphi(z)dz \right| \leq M \cdot 2\pi\rho,$$

so taking the limit as $\rho \to 0$ shows that

$$\int_C \varphi(z)dz = 0.$$

Consequently, as

$$\int_\Gamma \varphi(z)dz = \int_C \varphi(z)dz,$$

we have proved that

$$\int_\Gamma \varphi(z)dz = \int_\Gamma \frac{f(z) - f(z_0)}{z - z_0} dz = 0,$$

but this result is equivalent to

$$\int_\Gamma \frac{f(z)}{z - z_0} dz = f(z_0) \int_\Gamma \frac{dz}{z - z_0} = 2\pi i f(z_0),$$

and the theorem is proved. ■

Remark The Cauchy integral formula shows how a function $f(z)$ that is defined and analytic on a contour Γ defines $f(z)$ at *every* point inside Γ.

EXAMPLE 14.14 Find

$$\int_\Gamma \frac{\sinh z}{z^2 + (\pi/2)^2} dz,$$

where the contour Γ contains the point $z = i\pi/2$ but excludes the point $z = -i\pi/2$, and integration around Γ is in the positive sense.

Solution The integrand can be written

$$\frac{\sinh z}{z^2 + (\pi/2)^2} = \frac{\sinh z}{(z + i\pi/2)} \cdot \frac{1}{(z - i\pi/2)},$$

and because of the exclusion of the point $z = -i\pi/2$ from inside Γ, the function $\sinh z/(z + i\pi/2)$ is analytic inside Γ. Setting $f(z) = \sinh z/(z + i\pi/2)$ in the Cauchy integral formula with $z_0 = i\pi/2$, and integrating around Γ in the positive sense, gives

$$\int_\Gamma \frac{\sinh z}{z^2 + (\pi/2)^2} dz = \int_\Gamma \frac{f(z)}{z - i\pi/2} dz = 2\pi i f(i\pi/2)$$

$$= 2\pi i \cdot \frac{\sinh(i\pi/2)}{i\pi} = 2i \sin(\pi/2) = 2i. \qquad \blacksquare$$

The second Cauchy integral formula determines the derivatives of an analytic function in terms of a contour integral around a domain in which the function is analytic. The theorem can be stated as follows.

THEOREM 14.8

The Cauchy integral formula for derivatives Let $f(z)$ be a single-valued analytic function in a simply connected domain D containing a contour Γ in the form of a simple closed curve. Then, for any point z_0 inside Γ,

expressing $f^{(n)}(z)$ as an integral

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad \text{for } n = 1, 2, \ldots.$$

Proof The result follows by differentiating the statement of Theorem 14.7 with respect to z_0, and this in turn involves justifying differentiating under a contour integral sign. To simplify the proof of the Cauchy integral theorem for derivatives, this operation will be assumed to be justified, and an outline proof of its legitimacy will be postponed until the end of this section.

Let us consider the function $\varphi(\zeta, z) = f(\zeta)/(\zeta - z)$ to be a function of the two complex variables z and z_0. Differentiation of the result of Theorem 14.7 with respect to z_0 gives

$$f'(z) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

so, if we assume differentiation under the integral sign is permissible, this becomes

$$f'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\partial}{\partial z} \left(\frac{f(\zeta)}{\zeta - z} \right) d\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

and the result has been established for $n = 1$. The result for $n > 1$ follows by using mathematical induction, so the theorem is proved. $\blacksquare$

EXAMPLE 14.15

Find the value of the integral

$$\int_\Gamma \frac{\cos z}{(z - \pi/4)^3} dz,$$

where integration is in the positive sense around the circle Γ given by $|z - \pi/2| = 1$.

Solution Matching the integrand to the one in Theorem 14.8 shows that $f(z) = \cos z$, $n = 2$, and $z_0 = \pi/4$, so z_0 lies inside Γ. As $f^{(2)}(z) = -\cos z$, substitution into

the Cauchy integral formula for derivatives gives

$$\frac{2!}{2\pi i}\int_\Gamma \frac{\cos z}{(z-\pi/4)^3}dz = f^{(2)}(\pi/4) = -\frac{1}{\sqrt{2}},$$

showing that

$$\int_\Gamma \frac{\cos z}{(z-\pi/4)^3}dz = -\frac{i\pi}{\sqrt{2}}.$$

The next result has far-reaching consequences, because it says that an analytic function can be differentiated arbitrarily many times and the result will still be an analytic function.

THEOREM 14.9

an analytic function can be differentiated arbitrarily many times

An analytic function has derivatives of all orders A function $f(z)$ that is analytic in a simply connected domain D has derivatives of all orders.

Proof The result follows directly from Theorem 14.8. ■

A useful property of harmonic functions is stated in the next theorem, the proof of which makes use of the Cauchy–Riemann equations.

THEOREM 14.10

derivatives of harmonic functions are harmonic

Harmonic functions have partial derivatives that are harmonic A function $u(x, y)$ that is harmonic throughout a domain D has partial derivatives u_x, u_y, u_{xx}, u_{xy}, and u_{yy} that exist and are themselves harmonic functions.

Proof Around each point $z_0 = x_0 + iy_0$ inside D, construct a disc $|z - z_0| \le \rho$, all points of which lie in D. The Cauchy–Riemann equations can be used to construct a conjugate harmonic function v in the disc such that $f(z) = u + iv$ is analytic throughout the disc. From the Cauchy–Riemann equations we have $f'(z) = u_x + iv_x = v_y - iu_y$, but Theorem 14.8 asserts that $f'(z)$ is analytic in the disc, so the functions u_x and u_y must themselves be harmonic in the disc.

A repetition of this argument, coupled with the fact that $f''(z)$ is also analytic in the disc, establishes that u_{xx}, u_{xy}, and u_{yy} must be harmonic functions in the disc. By selecting a suitable choice of points z_0, each as the center of a disc with an appropriate radius ρ, it is possible to include all points of D in a set of overlapping discs. The result is true in each disc, so the theorem is proved. ■

We remark that the method used in Theorem 14.10 to extend the analytic function $f''(z)$ from the interior of disc C to the domain D, throughout which $f(z)$ is analytic, is called **analytic continuation**.

Further Results

The following is an outline proof of the legitimacy of the operation of differentiation under the integral sign with respect to a parameter. The result we obtain, known as **Leibniz' rule for analytic functions**, is a little more general than is necessary for the proof of Theorem 14.8.

THEOREM 14.11

Leibniz' rule—Differentiation under a contour integral Let $z = x + iy$ be a point on a simple closed curve Γ in a domain D, and let $z_0 = x_0 + iy_0$ be a point inside

Γ in which a function $g(z, z_0)$ is analytic with a continuous derivative $\partial g(z, z_0)/\partial z_0$ for all z and z_0. Then the function

$$G(z_0) = \int_\Gamma g(z, z_0)dz$$

is analytic in D and

$$G'(z_0) = \int_\Gamma \frac{\partial g(z, z_0)}{\partial z_0} dz.$$

Proof Write the functions $g(z, z_0)$ and $G(z_0)$ in the cartesian form

$$g(z, z_0) = u(x, y, x_0, y_0) + iv(x, y, x_0, y_0) \quad \text{and} \quad G(z_0) = U(x_0, y_0) + iV(x_0, y_0).$$

Then, as $G(z_0) = \int_\Gamma g(z, z_0)dz$, substituting for $g(z, z_0)$ in the integral we obtain

$$U(x_0, y_0) = \int_\Gamma udx - vdy \quad \text{and} \quad V(x_0, y_0) = \int_\Gamma vdx + udy.$$

As the partial derivatives of u and v are continuous with respect to all their dependent variables, it follows from real analysis that these last two real integrals can be differentiated under their integral signs with respect to x and y. Consequently,

$$\frac{\partial U}{\partial x_0} = \int_\Gamma \frac{\partial u}{\partial x_0}dx - \frac{\partial v}{\partial x_0}dy, \quad \frac{\partial V}{\partial x_0} = \int_\Gamma \frac{\partial v}{\partial x_0}dx + \frac{\partial u}{\partial x_0}dy,$$

with similar results for $\partial U/\partial y_0$ and $\partial V/\partial y_0$.

Using the Cauchy–Riemann equations, we can rewrite these results as

$$\frac{\partial U}{\partial y_0} = \int_\Gamma \frac{\partial u}{\partial y_0}dx - \frac{\partial v}{\partial y_0}dy = -\frac{\partial V}{\partial x_0} \quad \text{and, similarly,} \quad \frac{\partial U}{\partial x_0} = \frac{\partial V}{\partial y_0},$$

showing that U and V satisfy the Cauchy–Riemann equations in D. As the partial derivatives of U and V are continuous, it follows that $G(z_0)$ must be analytic in D. This proves the first part of the theorem. To prove the second part we use the fact that

$$G'(z_0) = \frac{\partial U}{\partial x_0} + i\frac{\partial V}{\partial x_0} = \int_\Gamma \left(\frac{\partial u}{\partial x_0} + i\frac{\partial v}{\partial x_0}\right)dx + \left(-\frac{\partial v}{\partial x_0} + i\frac{\partial u}{\partial x_0}\right)dy$$

$$= \int_\Gamma \left(\frac{\partial u}{\partial x_0} + i\frac{\partial v}{\partial x_0}\right)(dx + idy) = \int_\Gamma \frac{\partial g(z, z_0)}{\partial z_0}dz,$$

and the proof is complete. ■

GOTTFRIED WILHELM LEIBNIZ (1646–1716)
A German mathematician who studied moral philosophy and law, first at the University of Leipzig and then at the University of Altdorf, from where he obtained his degree. Declining an offer of a professorship at Altdorf, he embarked on a legal career and chose to develop his mathematical work as a personal interest. He traveled extensively, meeting distinguished people in many countries, including Isaac Newton, whom he met during a visit to the Royal Society of London. He published his work on the calculus about a decade after Newton had completed his own fundamental work on the calculus, but before its publication. It was due to Newton's cautious and suspicious nature that the publication of his work was delayed, leading to the long-standing international dispute over who should be considered to be the founder of the calculus. Shortly before his death Leibniz founded the Berlin Academy of Sciences.

Summary The Cauchy integral formulas were derived that express $f(z_0)$ and $f^{(n)}(z_0)$ in terms of integrals involving $f(z)/(z - z_0)^{n+1}$ around a contour containing z_0. Some important properties of analytic functions were obtained, and Leibniz' rule for differentiation under a contour integral was proved.

EXERCISES 14.3

In Exercises 1 through 8 use Theorem 14.7 to evaluate the given integral when integration is around Γ in the positive sense.

1. $\displaystyle\int_\Gamma \frac{\sin 2z}{z^2 - (\pi/2)^2}dz$, with Γ the circle $|z-1| = 1$.

2. $\displaystyle\int_\Gamma \frac{(1+z)e^z}{z^2 - 3z}dz$, with Γ the circle $|z| = 1$.

3. $\displaystyle\int_\Gamma \frac{\sin(\pi z/4)}{z^2 - 1}dz$, with Γ the circle $|z-1| = 1$.

4. $\displaystyle\int_\Gamma \frac{\cosh z}{z^2 + 1}dz$, with Γ the circle $|z-i| = 1$.

5. $\displaystyle\int_\Gamma \frac{e^z}{z - 4}dz$, with Γ the circle $|z-6| = 3$.

6. $\displaystyle\int_\Gamma \frac{(3+z^2)}{z\cosh z}dz$, with Γ the circle $|z| = 1$.

7. $\displaystyle\int_\Gamma \frac{z\sinh z}{z^2 + 1}dz$, with Γ the circle $|z+i/2| = 1$.

8. $\displaystyle\int_\Gamma \frac{\sin z}{z^2 + 1}dz$, with Γ the circle $|z-2i| = 2$.

In Exercises 9 through 15 use Theorem 14.8 to evaluate the given integral analytically when integration is around Γ in the positive sense, and verify the result by using computer algebra.

9. $\displaystyle\int_\Gamma \frac{z\sin z}{(z-\pi/4)^5}dz$, with Γ the circle $|z-\pi/4| = \pi$.

10. $\displaystyle\int_\Gamma \frac{z\cosh z}{(z-i)^4}dz$, with Γ the circle $|z-i| = 1$.

11. $\displaystyle\int_\Gamma \frac{\sin^2 z}{(z-\pi/2)^3}dz$, with Γ the circle $|z| = \pi$.

12. $\displaystyle\int_\Gamma \frac{\exp z^2}{(z+i)^4}dz$, with Γ the circle $|z+i| = 2$.

13. $\displaystyle\int_\Gamma \frac{z^2\sinh z}{(z-i)^4}dz$, with Γ the circle $|z| = 3$.

14. $\displaystyle\int_\Gamma \frac{(1-z)\cos z}{(z+i)^5}dz$, with Γ the circle $|z| = 2$.

15. $\displaystyle\int_\Gamma \frac{ze^z}{(z^2+1)^2}dz$, with Γ the circle $|z+2i| = 2$.

16. The Legendre polynomial $P_n(z)$ can be defined by the *Rodrigues formula* (Exercise 16, Section 8.2):

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n, \quad n = 0, 1, 2, \ldots.$$

Use the Cauchy integral formula for derivatives to show that

$$P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}dt,$$

where Γ is any simple closed curve containing the point $t = z$ in its interior, and integration is around Γ in the positive sense. This result is called the *Schläfli* contour integral representation of $P_n(z)$.

Further Results

The first exercise provides an upper bound for the modulus of the nth derivative of a function that is analytic in a disc, while the remaining exercises offer an introduction to the study of special functions and linear differential equations by means of contour integrals.

17.* Use the Cauchy integral formula for derivatives to prove that if $f(z)$ is an analytic function in a domain D containing a disc Γ of radius R with its center at $z = z_0$, and $|f(z)| \le M$ for all z on Γ, then

$$\left|f^{(n)}(z_0)\right| \le \frac{Mn!}{R^n}, \quad \text{for } n = 1, 2, \ldots.$$

These results are called the **Cauchy inequalities for derivatives**.

18.* Show, by considering the change in the argument of $(t^2-1)^{n+1}/(t-z)^{n+1}$ around a simple closed curve Γ with positive orientation that contains the point $t = z$, that

$$\int_\Gamma \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+1}}\right]dt = 0.$$

19.* Find the form taken by the result of Exercise 18 when the differentiation under the integral sign has been performed. Use the definition of $P_n(z)$ given in Exercise 16 to find $P_{n+1}(z)$, and by differentiation with respect to z find $P_n'(z)$ and $P_{n+1}'(z)$. Use these results in the first part of this exercise to derive the recurrence

relation

$$P'_{n+1}(z) = zP'_n(z) + (n+1)P_n(z).$$

20.* Show, by considering the change in the argument of $t(t^2-1)^n/(t-z)^n$ around a simple closed curve Γ with positive orientation that contains the point $t = z$, that

$$\int_\Gamma \frac{d}{dt}\left[\frac{t(t^2-1)^n}{(t-z)^n}\right] dt = 0.$$

21.* Find the form taken by the result of Exercise 20 when the differentiation under the integral sign has been performed. Use the definition of $P_n(z)$ in Exercise 16 to find $P_{n-1}(z)$ and $P_{n+1}(z)$, and use them in the result of the first part of the exercise to derive the recurrence relation

$$(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0.$$

22.* Show, by considering the change in the argument of $(t^2-1)^{n+1}/(t-z)^{n+2}$ around a simple closed curve Γ with positive orientation that contains the point $t = z$, that

$$\int_\Gamma \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right] dt = 0.$$

23.* Differentiate the integral representation for $P_n(z)$ given in Exercise 16 with respect to z to find $P'_n(z)$ and $P''_n(z)$ and form the expression

$$G(z) = (1-z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z).$$

Show that

$$G(z) = \frac{(n+1)}{2^{n+1}\pi i}\int_\Gamma \frac{(t^2-1)^n}{(t-z)^{n+3}}[2(n+1)t(t-z) - (n+2)(t^2-1)]dt.$$

By comparing the integrand of $G(z)$ with the differentiated form of the integrand in Exercise 22, deduce that $G(z) = 0$, and hence show that $P_n(z)$ is a solution of the Legendre differential equation

$$(1-z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0.$$

24.* By integrating $\exp(-z^2)$ around the rectangle with its corners at the points $(0, 0)$, $(R, 0)$, (R, b) and $(0, b)$ in the complex plane, proceeding to the limit as $R \to \infty$, and using the standard result $\int_0^\infty \exp(-x)^2 dx = \frac{1}{2}\sqrt{\pi}$, show that

$$\int_0^\infty \exp(-x)^2 \cos(2ax)dx = \frac{1}{2}\sqrt{\pi}\exp(-a^2).$$

Find the value of $\int_0^\infty \exp(-x)^2 \sin(2ax)dx$ in terms of a.

14.4 Some Properties of Analytic Functions

The next group of theorems describe some of the most important properties of analytic functions that can be deduced either directly or indirectly from the Cauchy integral theorem.

The first result, known as *Morera's theorem*, is the converse of the Cauchy–Goursat theorem and it is largely of theoretical importance.

THEOREM 14.12

Morera's theorem If a function $f(z)$ is continuous in a domain D and such that

$$\int_\Gamma f(z)dz = 0$$

for every simple closed contour Γ in D, then $f(z)$ is analytic in D.

Proof The condition

$$\int_\Gamma f(z)dz = 0$$

implies that the function

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta,$$

with z, z_0 and Γ in D, is independent of the path from z_0 to z. The continuity of $f(z)$ implies that $F(z)$ is differentiable, and from the argument preceding Theorem 14.5 it follows that $F(z)$ is analytic, with $F'(z) = f(z)$. Consequently, as $f(z)$ is the derivative of an analytic function, $f(z)$ must be analytic in D, so the theorem is proved. ■

The next result to be established is **Liouville's theorem** and it has numerous applications, one of which will occur later in the proof of the *fundamental theorem of algebra*.

THEOREM 14.13

Liouville's theorem If $f(z)$ is analytic in the entire z-plane, and such that $|f(z)| \leq M$ for all z, then $f(z) = \text{constant}$.

Proof Setting $n = 1$, $z - z_0 = Re^{i\theta}$, and $dz = iRe^{i\theta}d\theta$ in the Cauchy integral formula for derivatives and taking the modulus gives

$$|f'(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z)|}{|z - z_0|^2}|iRe^{i\theta}|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{R^2}Rd\theta,$$

and so

$$|f'(z_0)| \leq \frac{M}{R},$$

which is true for all z_0 independently of R. Taking the limit as $R \to \infty$ and dropping the suffix zero show that $|f'(z)| = 0$ for all z, but this is only possible if $f'(z) \equiv 0$, so $f(z) = \text{constant}$ and the result is proved. ■

Liouville's theorem illustrates one of the major differences between analytic functions in compex analysis and and differentiable functions in real analysis, because the theorem has no analogue in real analysis. This is easily seen by considering the function $\sin x$, which, although differentiable, bounded, and defined for all x, is not a constant. Another important difference between analytic functions and real functions is that a real function may only be differentiable a finite number of times, whereas analytic function has derivatives of all orders.

The next theorem is used repeatedly when seeking the zeros of polynomials, and it is proved here for a general complex polynomial of degree n.

THEOREM 14.14

every polynomial of degree *n* has *n* zeros

Fundamental theorem of algebra Every complex polynomial $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, with complex coefficients $a_0, a_1, \ldots, a_n$ with $a_n \neq 0$ and $n \geq 1$ has precisely n zeros, some of which may be repeated.

Proof The proof will be by contradiction. Suppose, if possible, that $P_n(z)$ has no zeros. Then the function $Q_n(z) = 1/P_n(z)$ is analytic for all z (it is an *entire function*). Then, when $|z|$ is large, $|P_n(z)|$ can be approximated by $|P_n(z)| \approx |a_n z^n|$, so it follows that $\lim_{|z|\to\infty} |Q_n(z)| = \lim_{|z|\to\infty} 1/|P_n(z)| = 0$. Consequently, $|Q_n(z)|$ is bounded in the entire complex plane, so by Liouville's theorem $Q_n(z)$ must be a constant. This contradicts the definition of $Q_n(z)$, showing that $P_n(z)$ must have at least one zero.

Denoting this zero by z_1, we can remove a factor $(z - z_1)$ from $P_n(z)$ and write it as $P_n(z) = (z - z_1)P_{n-1}(z)$, where $P_{n-1}(z)$ is a polynomial of degree $n - 1$. This process of factoring out $(z - z_0)$ from $P_n(z)$ to arrive at the polynomial $P_{n-1}(z)$ of lower degree is called **deflation**. Applying the same form of argument to $P_{n-1}(z)$ proves the existence of another zero z_2, and repetition of the argument establishes the existence of precisely n zeros, not all of which need be different. ■

THEOREM 14.15

an averaging property for analytic functions

Gauss's mean value theorem Let $f(z)$ be analytic in a simply connected domain D containing the circle Γ of radius ρ with its center at z_0. Then,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta.$$

Proof From the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz,$$

but on the circle $z - z_0 = \rho e^{i\theta}$ and $dz = i\rho e^{i\theta}d\theta$, so

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta.$$ ■

When expressed in words, the Gauss mean value theorem says that the value of an analytic function $f(z)$ at a point z_0 in D is the average of the values of $f(z)$ around the perimeter of any circle Γ in D with its center at the point z_0. A useful consequence of this theorem is the following result for harmonic functions that we state in the form of a corollary.

COROLLARY TO THEOREM 14.15

an averaging property for harmonic functions

Mean value theorem for harmonic functions Let $u(x, y)$ be harmonic in a domain D containing the point $z_0 = x_0 + iy_0$, and let Γ be any circle of radius ρ in D with its center at (x_0, y_0). Then $u(x_0, y_0)$ is the average of the values of $u(x, y)$ around the perimeter of Γ.

Proof The corollary follows immediately from Theorem 14.15 by setting $f(z) = u + iv$ and equating the real parts of the statement of the theorem. ■

THEOREM 14.16

A function with its maximum modulus at the center of a disc Let a function $f(z)$ be analytic in a disc with its center at the point z_0, and continuous on its circular boundary Γ. Then if the modulus $|f(z)|$ attains its maximum value M at z_0, the function $f(z)$ = constant throughout the disc and on its boundary Γ.

Proof The proof of the theorem contains two steps. The first involves showing that the conditions of the theorem lead to the result that $|f(z)| = M$ inside the disc and on its boundary Γ. The second step that completes the proof involves showing that a function with constant modulus that is analytic in a disc must, of necessity, be constant.

STEP 1 Let the function $f(z)$ be analytic inside the circle $z = z_0 + \rho e^{i\theta}$ and continuous on its boundary Γ, and let its modulus $|f(z)|$ attain its maximum value $M > 0$ at z_0. Suppose, if possible, that $|f(z)| < M$ at some point on Γ. Then because the function is continuous on Γ, it must follow that $|f(z)| < M$ over some finite part of Γ.

From the Gauss mean value theorem,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta,$$

so if we take the modulus, this becomes

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta.$$

As $|f(z_0)| = M$, this becomes

$$M \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta.$$

However, the integrand is less than M over some part of Γ, so for some k such that $0 < k < 1$,

$$\int_0^{2\pi} \left|f\left(z_0 + \rho e^{i\theta}\right)\right| d\theta = kM.$$

Using this result with the previous one leads to the equation

$$M \le kM \quad (0 < k < 1),$$

but this result is impossible, so $k = 1$ and $|f(z)| = M$ on Γ. As the disc is of radius ρ the result will be true for any radius r such that $0 \le r \le \rho$, and we have proved that $|f(z)| = M$ inside and on the boundary of the disc.

STEP 2 Setting $f(z) = u + iv$ we can write $|f(z)|^2 = u^2 + v^2$, so from the result of Step 1 we see that $u^2 + v^2 = M^2$ throughout the disk. Differentiating this result partially with respect to x and y gives

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

The Cauchy–Riemann equations then allow these equations to be rewritten as

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0.$$

Solving these equations for u_x and u_y gives $(u^2 + v^2)u_x = 0$ and $(u^2 + v^2)v_y = 0$, but $u^2 + v^2 = M^2 > 0$, so the only solution of this system of equations is $\partial u/\partial x = \partial u/\partial y = 0$, showing that $u =$ constant. Using $u =$ constant in the Cauchy–Riemann equations implies that $\partial v/\partial x = \partial v/\partial y = 0$, so $v =$ constant, and we have shown that $f(z) =$ constant throughout the disc. The proof is complete. ∎

THEOREM 14.17

an extremum principle for $|f(z)|$ when $f(z)$ is analytic

The maximum/minimum modulus principle If a nonconstant function $f(z)$ is analytic in a bounded domain D and continuous on its boundary Γ, then the maximum of $|f(z)|$ must occur on Γ. If $f(z) \neq 0$ anywhere in D, then the minimum value of $|f(z)|$ also must occur on Γ.

Proof The conditions that $f(z)$ is analytic in D and continuous on Γ imply that $f(z)$ is continuous throughout D and on its boundary Γ. Consequently, the real function $|f(z)|$ must have both a maximum and a minimum in the closed region formed by D and its boundary Γ.

As $f(z)$ is analytic in D, so also is $[f(z)]^n$ for $n = 2, 3, \ldots$, so taking a point z_0 inside D and applying the Cauchy integral theorem to $[f(z)]^n$ gives

$$[f(z_0)]^n = \frac{1}{2\pi i}\int_\Gamma \frac{[f(z)]^n}{z - z_0}dz.$$

If $|f(z)| \le M$ on the boundary Γ of finite length L, and if d is the minimum distance from z_0 to Γ, taking the modulus of this result we obtain

$$\begin{aligned} |f(z_0)|^n &\le \frac{1}{2\pi}\int_\Gamma \frac{|f(z)|^n}{|z - z_0|}|dz| \\ &\le \frac{1}{2\pi}\frac{M^n L}{d}, \end{aligned}$$

showing that

$$|f(z_0)| \le M\left(\frac{L}{2\pi d}\right)^{1/n}.$$

Proceeding to the limit as $n \to \infty$ leads to the result $|f(z_0)| \le M$, so the value of $|f(z)|$ throughout the domain cannot exceed its maximum value on the boundary Γ.

To complete the proof suppose, if possible, that in addition to the maximum value of the modulus occurring on the boundary, it also occurs at a point z^* inside D. Construct a circle inside D with z^* as its center. Then from Theorem 14.16 the function $f(z)$ must be constant inside this circle. As the $f(z)$ is analytic in D, it is also continuous in D together with all its derivatives, and, in particular, it is continuous across the boundary of the circle. The derivatives of $f(z)$ are zero inside and on the boundary of the circle, so by continuity they must also be zero throughout the rest of D, from which it follows that $f(z) =$ constant in D. This contradicts the assumption that $f(z)$ is nonconstant, so the maximum value of $|f(z)|$ can only occur on the boundary Γ.

The minimum value of $|f(z)|$ must also occur on the boundary Γ if $f(z) \ne 0$ in D, because if the foregoing result is applied to the function $\varphi(z) = 1/f(z)$, the maximum value of $|\varphi(z)|$ must occur on the boundary of Γ, but this corresponds to the minimum value of $|f(z)|$, so the theorem is proved, because if $f(z) = 0$ in D then $1/f(z)$ is not analytic in D. ■

EXAMPLE 14.16

Confirm by direct calculation the maximum/minimum principle for the function $f(z) = \sin z$ in the domain D defined by $0 \le x \le \pi$ and $0 \le y \le 1$, and place bounds on $|\sin z|$ inside D.

Solution We notice first that the function $f(z)$ is analytic for all z and the domain D is bounded. Setting $z = x + iy$ in $f(z)$ and expanding the result gives $\sin z = \sin x \cosh y + i \cos x \sinh y$, from which it follows that

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y.$$